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Arthur Gibadullin
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Preface

The conference was held to summarize international experience in the field of mathematics, information, and computing technologies. The III International Seminar on Computing Technologies and Applied Mathematics was held on October 07–11, 2024, in person at Komsomolsk-na-Amure State University (Komsomolsk-on-Amur, Russia). The conference addressed issues of mathematics related to mathematical and optimization modeling, numerical algorithms, and high-performance computing and applications. A distinctive feature of the conference is that authors from Germany, China, South Korea, Uzbekistan, and Russia spoke at it. Researchers from different countries presented scientific and practical proposals on mathematical modeling and computing technologies.

The editors of the conference proceedings are Arthur Gibadullin from the National Research University "MPEI," Moscow, Russia, and Sergey Gordin from Komsomolsk-na-Amure State University, Komsomolsk-on-Amur, Russia. Thus, the conference allowed developing new scientific recommendations on the use of mathematics, information, computer, digital and intelligent technologies, and networks in various fields of activity, which can be useful for state and regional authorities, international and supranational organizations, and scientific and professional community.

Each presented paper was reviewed by at least three members of the Program Committee or an independent reviewer. As a result of the work of all reviewers, 14 articles out of 37 received materials were accepted for publication. The reviews were based on the assessment of the topic of the presented materials, relevance of the research, scientific significance and novelty, quality of materials, and originality of the work. Reviewers, members of the Program Committee, and members of the Organizing Committee did not enter into discussions with the authors of the articles.

The Organizing Committee of the conference expresses its gratitude to the Springer staff that supported the publication of this collection. In addition, the Organizing Committee thanks the conference participants, reviewers, and everyone who helped organize this conference and form this volume for publication in the Springer Proceedings in Mathematics & Statistics series.

Moscow, Russia Arthur Gibadullin
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Contents

Refinement of Variable Order Fractional Derivative of Gerasimov-Caputo Type by Multidimensional Levenberg-Marquardt Optimization Method

Dmitriy Tverdyi

Abstract When solving many fundamental and applied problems related to the study of properties of dynamical systems, situations often arise when the parameters of the model are not exactly defined. One can try to solve such a problem by searching the parameters, taking into account the understanding of the nature of the modelling process. On the other hand, it is possible to solve inverse problems—a common type of problems in many scientific fields, where it is necessary to determine the values of the model parameters on the basis of experimental data, but it is impossible to make direct measurements of the parameters. The paper considers an inhomogeneous fractional equation with a Gerasimov-Caputo type operator of variable $\alpha(t)$ order. The possibility of recovering the form of the function $\alpha(t)$ and refining its values on the basis of test generated data is investigated. The direct problem is defined as a Cauchy problem for a fractional equation solved numerically. The solution of the inverse problem is reduced to the minimization of the inviscid functional, and the minimization problem is solved using the iterative Levenberg-Marquardt unconditional optimization method. Thus, the problem of determining the optimal form $\alpha(t)$ of a fractional equation is reduced to 4 parameters that control the course of the solution of the inverse problem and do not depend on the physical meaning of the original problem in specific models. On test examples it was shown that the Levenberg-Marquardt method can indeed be used for unconditional optimization to determine type of function $\alpha(t)$ and its optimal values.

Keywords Inverse problems · Non-conditional optimization · Levenberg-Marquardt algorithm · Fractional derivatives · Gerasimov-Caputo · Memory effect · Non-locality · Implicit finite-difference schemes

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1 Introduction

Integral-differential calculus has been actively developing for more than 300 years, and during this time it has proved to be a reliable and powerful tool for solving theoretical and applied problems. Throughout the history of this branch of mathematical science, researchers have been fascinated by the question of the possibility of generalizing the concepts of integral and derivative from integer to real order [1]. Since the time of G. W. Leibniz and G. F. Lopital (1965), various definitions of the concept of fractional differentiation operator have been developed: Riemann-Liouville, Weyl, Grunwald-Letnikov and many others. Also, applications of fractional derivatives and integrals are increasingly being found in areas where the integer definition was insufficient. In quantum physics [2], viscoelasticity [3, 4] and solid mechanics [5], anomalous diffusion processes [6], fractal oscillators [7], etc.

In our study, fractional derivatives (FD) are interesting because they are useful for illustrating the memory and hereditary properties of many dynamical processes [8]. This effect is also called hereditary, and one of the first scientists to work on deriving this theory was the Italian mathematician and physicist V. Volterra [9]. Hereditary and its relation to fractional derivatives can be shown as follows. Consider the following integro-differential equation (IDE):

$$\int_{-\infty}^{t} K(t-\sigma)\dot{u}(\sigma)d\sigma = F(u(t),t), \tag{1}$$

where, according to V. Voltera [9] principles of heredity, memory takes into account all possible prehistory. Since in practice we do not have data on the entire history of the process, we should consider the integral on the sub-interval $(0, t)$ where, $t \in [0, T]$ is the current simulation time, $T > 0$ is the total simulation time. The main point is that choosing the form of the kernel $K(t - \sigma)$ as the degree function $K(t-\sigma) = \frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}$ allows us to pass to the fractional calculus operators [7, 10].

Turning to the FD formalism when dealing with such IDE, in our study we will consider the integral in the left-hand side of (1) in terms of the Gerasimov-Caputo fractional differentiation operator [11, 12] of constant $0 < \alpha < 1$ order:

$$\int_{0}^{t} \left(\frac{(t-\sigma)^{-\alpha}}{\Gamma(1-\alpha)}\right)\dot{u}(\sigma)d\sigma \equiv \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t} \frac{\dot{u}(\sigma)}{(t-\sigma)^{\alpha}}d\sigma \equiv \partial_{0,t}^{\alpha}u(\sigma), \tag{2}$$

where, $\Gamma(\cdot)$ is the Euler gamma function; $\dot{u} = du/dt$.

The memory effect is represented by the dependence of the current state of the dynamical system on the previous states, i.e. non-locality. The intensity of this effect will be determined by the value of the degree of FD. Classically, this certain α value is non-integer and constant [13].

In science and engineering, memory properties are common in complex systems. However, there are generalizations of fractional derivatives to the case of time-variable non-locality $\alpha(t)$ and other functional dependencies [14]. Variable order fractional derivatives (VOFD) find their application in the theory and practice of physical and mathematical sciences. For example, the diffusion equations [15] or the dynamic viscoelastic oscillator model [16].

In [17] generalization FD of the form (2) to VOFD at $0 < \alpha(t) < 1$ studied in detail:

$$\frac{1}{\Gamma(1-\alpha(t))}\int_0^t \frac{\dot{u}(\sigma)}{(t-\sigma)^{\alpha(t)}}\,d\sigma \equiv \partial_{0,t}^{\alpha(t)}u(\sigma), \tag{3}$$

where, $\alpha(t) \in C^1(0,1)$ is a continuous-differentiable function.

In such a case, the Eq. (1) in the domain $\Omega = \{(t) : 0 \le t \le T\}$ can be represented as a fractional equation (FDE) of the form:

$$\partial_{0,t}^{\alpha(t)}u(\sigma) = F(u(t),t), \tag{4}$$

where, $u(t) \in C^2(0,T)$ a twice continuously differentiable solution function; $F(u(t),t)$ a functional defining the form of the FDE.

The series of works of the author of this research, including publications [18, 19], is devoted to the study of issues related to mathematical modeling of volumetric activity of radon gas (RVA) in accumulation chambers with gas-discharge sensors. Accumulation chambers are located in observation points, which are located above the zones of dynamic influence of faults in the Earth's crust. These zones have increased permeability, which favors the flow of subsurface gases into the atmosphere [20]. It is assumed that the parameter $\alpha(t)$ describes the fractality of the geosphere and is related to its characteristics such as porosity, permeability and fracturing, which directly affect the intensity of the process. As a result, a number of mathematical models based on nonlinear FDE of the form (4) have been developed. In [18] it is shown that nonlinear models with FD of the form (2) of constant order α can well describe the accumulative modes of RVA. In [18] it is shown that generalization to VOFD of the form (3) to the case of $\alpha(t)$ also allows describing some anomalous RVA modes. Continuous monitoring of RVA variations in order to detect anomalies in its values is one of the effective methods to investigate the stress-strain state of the geosphere. RVA is considered to be an informative and operative precursor of seismic events, which determines the relevance of such studies.

When solving many fundamental and applied problems related to the study of certain properties of dynamical systems, situations often arise when the parameters of the model are not precisely determined. However, if experimental data or information on the ranges of values of the model parameters are known, it is possible to select various parameters of such models empirically, taking into account the available understanding of the nature of the modeled process. In the [18, 19] articles, the model parameters are refined by enumerating values and comparing time series: modeling results and experimental data representing the process. This process continues until the modelling results begin to qualitatively approximate the data, that is, until high values of correlation and R^2 determination [21] are reached. This approach is time consuming, which inevitably leads us to ideas of different ways to automate the selection of optimal parameters.

For example, in the research [22, 23] for fractional equations, numerical methods for obtaining interval estimates of solutions from known interval values of parameters for anomalous diffusion problems are applied.

An inverse problem can be solved to obtain a solution to the problem of finding the optimal parameters. Such problems are often ill-posed [24], i.e., the solution may not exist, the solution may not be unique, or the solution is unstable [25]. However, the inverse problem is a common type of problem in many scientific fields [26], where it is necessary to determine the values of model parameters from observed data [27]. Despite the fact that this branch of mathematics began to be actively developed only in the XIX-XX century, it can rightly be considered one of the most important. The need for such an approach often arises in geological data [29], in geophysics and seismology [30], in computed tomography [31, 32] and many others. This issue is especially acute for geophysics, as it is impossible to make direct measurements deep underground [27].

2 Direct Problem Statement

Consider the case when the functional $F(u(t), t) = 1$. Then (4) is defined as an inhomogeneous FDE for which in the domain $\Omega = \{(t) : 0 \le t \le T\}$ we consider a Cauchy problem of the form:

$$\partial_{0,t}^{\alpha(t)} u(\sigma) = 1, \quad u(0) = u_0, \tag{5}$$

where, $\partial_{0,t}^{\alpha(t)} u(\sigma)$—FD of variable order $0 < \alpha(t) < 1$ of the form (3); $u(t)$—solution function; $\alpha(t)$—degree of the fractional derivative of (4); $u(t) \in C^2(0, T) = U$—the class of twice continuously differentiable functions; $\alpha(t) \in C(0, 1) = A$—class of continuous-differentiable functions; u_0—some constant, the initial condition of the Cauchy problem.

Definition 1 The direct problem is a Cauchy (5) problem consisting of finding $u(t) \in U$ given a known $\alpha(t) \in A$.

To solve the direct problem (5), we use the previously developed and theoretically justified non-local implicit finite-difference scheme (IFDS), which has been tested on a number of test problems [17]. We also note that (5) can be solved using the explicit finite-difference scheme (EFDS) presented in [17, 28]. Let a uniform grid domain $\widehat{\Omega}$ be given, in which classes of grid functions $\widehat{U}$ and $\widehat{A}$ with a sampling step τ are defined:

$$\begin{aligned}&\tau = T/N, \widehat{\Omega} = (t_i = i\tau) : 0 \le i < N, \widehat{U} \in \widehat{\Omega}, \widehat{A} \in \widehat{\Omega},\\ &u(t) = u(t_i) = u_i, \alpha(t) = \alpha(t_i) = \alpha_i, 0 < \alpha_i < 1,\end{aligned} \tag{6}$$

The in-homogeneous FDE (5) can be approximated in the domain (6) by the IFDS scheme as follows:

$$\begin{aligned}&A_i \sum\nolimits_{j=0}^{i-1} w_j^i \left(u_{i-j} - u_{i-j-1}\right) - 1 = 0, u_0 = C,\\ &A_i = \frac{\tau^{-\alpha_i}}{\Gamma(2-\alpha_i)}, w_j^i = (j+1)^{1-\alpha_i} - j^{1-\alpha_i}, 1 \le i < N,\end{aligned} \tag{7}$$

where C—is a known constant.

Definition 2 The difference direct problem (7) is to find the grid $u(t_i) \in \widehat{U}$ in the domain (6) for known values of the grid function $\alpha(t_i) \in \widehat{A}$.

The IFDS scheme (7) can be solved by Newton's method or by a modified Newton's method. In [17], a number of theorems on the convergence and stability of IFDS for FDEs of the more general form (4) as well as fractional derivative approximation (3) were formulated and proved. More details on the numerical method, see in [17, 33].

Remark 1 The difference direct problem (7) based on the IFDS scheme and solved by Newton's method, according to the [17], is unconditionally stable and converges with order $O\left(\tau^{2-max\,1_t(\alpha(t))}\right)$.

3 Inverse Problem Statement

Let the values of the function $\alpha(t) \in A$ (or its grid analogue $\alpha(t_i) \in \widehat{A}$) be unknown. However, additional information (experimental data) is known about the solution of the difference direct Cauchy problem (7) in the domain $\widehat{\Omega}$:

$$u(t_i) = \theta(t_i), \tag{8}$$

Let $\alpha(t) \in A$ be a function of some known class, and its form is clearly defined by some set of parameters. Then, according to the book [34] by Tikhonov A.N., Samarskii A.A., the solution of the inverse problem is reduced to the search for the values of these parameters. Following this approach, let us define the set of vectors $\overrightarrow{\mathbb{X}}$ as the space of solutions to the inverse problem (5). The vector $\overrightarrow{X} =$

$[X_0, \ldots, X_{K-1}]$ a set of unknown parameters characterizing the form of the function $\alpha(t)$, where $\vec{X} \in \vec{\mathbb{X}}$, $\vec{\mathbb{X}} \subset \mathbb{R}^K$, K is the number of components.

Definition 3 The inverse problem for (5) is defined as the recovery (definition) of the function $\alpha(t) = \alpha\left(\vec{X}\right) \in A$:

$$\partial_{0,t}^{\alpha\left(\vec{X}\right)} u(\sigma) = 1, u(0) = \theta_0, \tag{9}$$

Definition 4 Therefore, the difference inverse problem for (7) is defined as the restoration of the grid function $\alpha(t_i) = \alpha\left(\vec{X}\right) = \alpha(X_0, \ldots, X_{K-1}) \in \widehat{A}$, from the known (8) additional information:

$$\begin{gathered} A_i \sum_{j=0}^{i-1} w_j^i \left(u_{i-j} - u_{i-j-1}\right) - 1 = 0, u_0 = \theta_0, u_i = \theta_i, \\ A_i = \frac{\tau^{-\left(\alpha\left(\vec{X}\right)\right)}}{\Gamma\left(2 - \left(\alpha\left(\vec{X}\right)\right)\right)}, w_j^i = (j+1)^{1-\left(\alpha\left(\vec{X}\right)\right)} - j^{1-\left(\alpha\left(\vec{X}\right)\right)}, 1 \le i < N, \end{gathered} \tag{10}$$

Let $\vec{\theta} = [\theta_0, \ldots, \theta_{N-1}]$ the vector of experimental data according to (8), and $\omega(t_i) \in \widehat{U}$, $\omega(t_i) = \omega\left(\vec{X}\right) = [\omega_0, \ldots, \omega_{N-1}]$ is the vector of model data, i.e., the solution of the difference direct problem (7) obtained with respect to the approximation $\alpha\left(\vec{X}\right)$ at some given values of $\vec{X}$.

Let $\vec{\eta} = \eta\left(\vec{X}\right) = [\eta_0, \ldots, \eta_{N-1}]$ be a bias vector of dimension $N > K$ such that $\mathbb{R}^K \to \mathbb{R}^N$, i.e., a real subset of $\mathbb{R}$ that depends on the solution $\omega\left(\vec{X}\right)$ of dimension N and hence depends on $\alpha\left(\vec{X}\right)$ of K parameters of $\vec{X}$:

$$\vec{\eta} = \vec{\theta} - \omega\left(\vec{X}\right), \tag{11}$$

Then, in terms of unconditional optimization theory, according to [35], the solution of the difference inverse problem (10) reduces to the minimization of the Ψ of bias functional:

$$\begin{gathered} \min_{\vec{X} \in \mathbb{R}^K} \Psi\left(\vec{X}\right) = M\left(\eta\left(\vec{X}\right)\right), \eta : \mathbb{R}^K \to \mathbb{R}^N, M : \mathbb{R}^N \to \mathbb{R}, \\ M\left(\eta\left(\vec{X}\right)\right) = \frac{1}{2} \vec{\eta}^T \vec{\eta} = \frac{1}{2} \sum_{i=0}^{N-1} \eta_i^2 = \frac{1}{2} \sum_{i=0}^{N-1} (\theta_i - \omega_i)^2, \end{gathered} \tag{12}$$

Remark 2 Since the continuous inverse problem is discretized, it often happens that exactly the stability condition obtained by numerical method is broken, defining the difference inverse problem (10) as an ill-posed [36]. However, in the considered case, the unconditionally stable IFDS algorithm [17] is used to solve the direct problem (5) in the discrete formulation (7).

4 Method of Minimizing the Target Function

To solve such a minimization problem (12), we turn to mathematical optimization methods [35, 37]. There is a class of methods called iterative descent or gradient descent methods [38, 39], but they have an unpleasant peculiarity. In practice, the values of $\vec{\eta}$ may have local minimums in addition to the desired global minimum. Depending on the choice of the initial approximation $\overrightarrow{X^0}$, the method may converge just to a local minimum of the functional $\Psi\left(\vec{X}\right)$ rather than a global minimum.

As an alternative to iterative descent methods, there is a class of Newtonian (quasi-Newtonian) methods that are among the most efficient in practice [35]. We can use Newton's unconditional optimization method, but it requires that the functional $\Psi\left(\vec{X}\right) \in C^2\left(G \subset \mathbb{R}^K\right)$ twice continuous-differentiable in an open convex set G. More details in [35]. This condition is required to compute the Hessian (Hesse matrix) of the form:

$$H = \nabla^2\Psi\left(\vec{X}\right) = \nabla^2\Psi\left(\vec{X}\right)_{i,k} = \frac{\partial^2\Psi\left(\vec{X}\right)}{\partial X_i \partial X_k}, \quad i \geq 0, \quad k \leq K-1,$$

Newton's unconditional optimization method is implemented as an iterative procedure:

$$\overrightarrow{\Delta X} = \left(-H^{-1}\right) \times \left(\nabla\Psi\left(\vec{X}\right)\right),$$

where, $\nabla\Psi\left(\vec{X}\right)$—the gradient of Ψ in $\vec{X}$; approximation $\vec{X}$—the recovered values computed at the current iteration of the procedure; $\overrightarrow{\Delta X} = [\Delta X_0 \ldots \Delta X_{K-1}]$—the optimal increments $\vec{X}$ for the next iteration.

Remark 3 An important characteristic of numerical methods of optimization is the convergence rate, which characterizes the efficiency of the method used to solve the inverse problem. Therefore, it is desirable to use the method with the highest convergence rate.

Therefore, we will use a modification of Newton's method, called the Levenberg-Marquardt iterative method of unconditional optimization [40–42], which allows us to get rid of the condition for the existence of second derivatives. Assuming that at least $\Psi\left(\vec{X}\right) \in C^1\left(G \subset \mathbb{R}^K\right)$ is satisfied, and $\nabla\Psi\left(\vec{X}\right) = J^T \times \vec{\eta}$, and the Hesse matrix of dimension $K \times K$ is calculated as follows:

$$H = J^T \times J + \gamma E,$$

then the Levenberg-Marquardt method is realized as the following iterative procedure for the system of linear algebraic equations:

$$\overrightarrow{\Delta X} = \left(- \left(J^T \times J + \gamma E \right)^{-1} \right) \times \left(J^T \times \vec{\eta} \right), \tag{13}$$

where,

- E—is a unitary matrix of dimension $K \times K$;
- $\vec{\eta} = \overrightarrow{\eta^{(n)}} = \eta\left(\overrightarrow{X^{(n)}}\right)$—bias vector;
- $J = J\left(\vec{X}\right) = J\left(\overrightarrow{X^{(n)}}\right)$—is a Jacobi matrix of dimension $N \times K$ whose elements have the form: $J_{i,k} = \frac{\partial \eta_i^{(n)}}{\partial X_k^{(n)}}, i = 0..N-1, k = 0..K-1$.

and the derivatives in $J_{i,\,k}$ are approximated by a finite difference of the form:

$$J_{i,k} = \frac{\eta_i^{(n)\delta} - \eta_i^{(n)}}{\delta X_k},$$

where,

- $\overrightarrow{\delta X} = [\delta X_0 \ldots \delta X_{K-1}]$—given small increments $\vec{X}$;
  $\overrightarrow{\eta^{(n)}} = \left[\eta_0^{(n)} \ldots \eta_{N-1}^{(n)}\right]$—is calculated by the formula (11) on n iterations of the method, based on the solution of the difference direct problem with respect to $\overrightarrow{X^{(n)}} = \left[X_0^{(n)}, \ldots, X_{K-1}^{(n)}\right]$ approximations;
- Moreover, $\overrightarrow{X^{(0)}}$ is the initial approximation for the components of $\vec{X}$;
- Similarly, we obtain $\overrightarrow{\eta^{(n)\delta}} = \left[\eta_0^{(n)\delta} \ldots \eta_{N-1}^{(n)\delta}\right]$ to solve the direct problem with respect to $\overrightarrow{X^{(n)\delta}} = \overrightarrow{X^{(n)}} + \overrightarrow{\delta X}$ approximations with given increments.

5 Algorithm of the Levenberg-Marquardt and Regularization

Remark 4 Typically, the so-called regularization parameter γ some number taking different values for different optimization problems, i.e., for different additional information $\vec{\theta}$.

The regularization parameter γ is of key importance for the Levenberg-Marquardt method and is directly related to the step and direction of convergence of the method. Therefore, it is worth noting important properties imposed on γ according to the:

$$\mathbb{Y} = \mathbb{R}_{>0} = \{\gamma | \gamma \in \mathbb{R}, \gamma > 0\};$$

- if $\gamma \in \mathbb{Y}$ and the Hesse matrix H is positively defined, then $\overrightarrow{\Delta X}$ is the direction of descent (gradient) for the "good" (optimal) step of the method;
- if $\gamma \rightarrow 0$ is very small, then $\overrightarrow{\Delta X}$ is the optimal step of the method in the gradient direction. This also means the convergence rate of the method is close to quadratic;
- If $\gamma \in \mathbb{Y}, \gamma \gg 0$, then it follows from (13) that $\overrightarrow{\Delta X} \simeq 1/ -{}^{1}\!/_{\gamma}\left(J^T \times \vec{\eta}\right)$ method step in the direction of the anti-gradient.

In this article we will not describe the Levenberg-Marquardt algorithm in detail; it can be found in detail in [42]. It is important to note only that there are two more parameters in the algorithm: c—is a given constant for recalculation of the regularization parameter and v—is a given constant, which is not necessary to compute the value of $\gamma^{(0)}$ at the start. The presented algorithm for solving the inverse problem (10) was implemented as a set of subroutines in C language.

Definition 5 As a result, the inverse problem for (4) is reduced to the choice of 4 parameters controlling the course of the iterative method, which are no longer tied to the physical meaning of the original task in specific models.

Remark 5 The peculiarity of the described methods of Newton-type minimization of the functional (12) for solving the difference inverse problem is that it is necessary to solve the difference direct problem (7) many times, different values of $\vec{X}$. This requires the most efficient implementations of the direct (7) problem and methods for solving it. We used the previously implemented subroutines [33] in C, using OpenMP [43] and/or CUDA [44], to speed up the calculations due to parallelization.

6 Test Examples and Results of Numerical Experiments

We will show by examples that the described method of solving the difference inverse problem (10), at certain given $\overrightarrow{X^{(0)}}$, $\delta X, c, v$ parameters related to the direction and step of the Levenberg-Marquardt method, really allows to restore the form of the functional dependence $\alpha(t)$. The structure of the numerical experiment is as follows:

Suppose for all examples further: $T = 30$, $N = 300$;

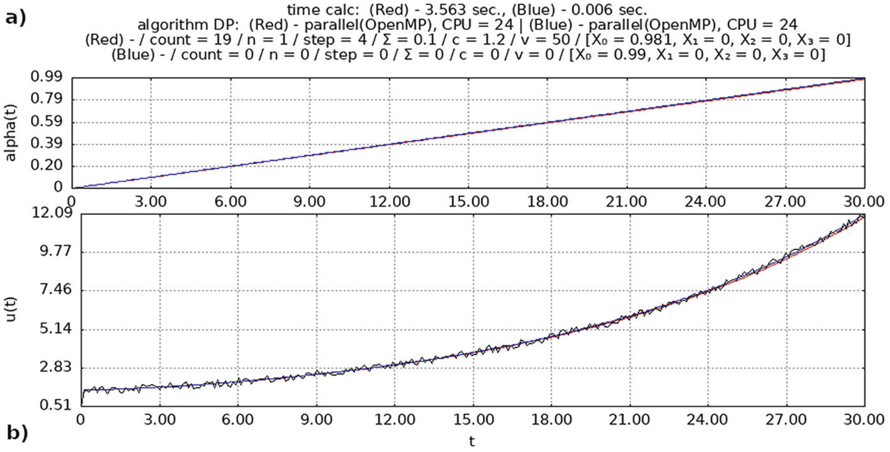

Fig. 1 Example 1. $\alpha(t_i) = X_0(i\tau/T)$—is a linear-increasing. (**a**) Graphs $\alpha(t_i)$: (Blue)—“benchmark” $\vec{X}$, (Red)—restored $\vec{X}$; (**b**) Model curves: (Blue)—“benchmark”, (Red)—optimization result at $\overrightarrow{X^{(0)}} = [0.01]$, $\delta X = [0.005]$

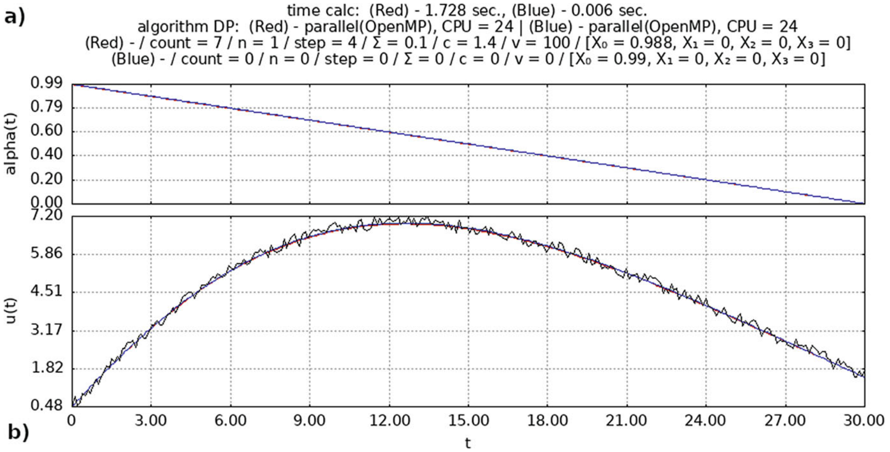

Fig. 2 Example 2. $\alpha(t_i) = X_0((N - i)\tau/T)$—linear-decreasing. (**a**) Graphs $\alpha(t_i)$: (Blue)—“benchmark” $\vec{X}$, (Red)—restored $\vec{X}$; (**b**) Model curves: (Blue)—“benchmark”, (Red)—optimization result at $\overrightarrow{X^{(0)}} = [0.01]$, $\delta X = [0.005]$

- Let us solve the “benchmark” direct problem (7) with all given and known parameters of the model, including the recovering function $\alpha(t)$ and its defining parameters $\vec{X} = [X_0, \ldots, X_{K-1}]$. Blue model curve in (Figs. 1, 2, 3, 4, and 5b);
- Based on the “benchmark” solution, we generate $\vec{\theta}$—pseudo-random experimental data (8). The (Black) curve in (Figs. 1, 2, 3, 4, and 5b);

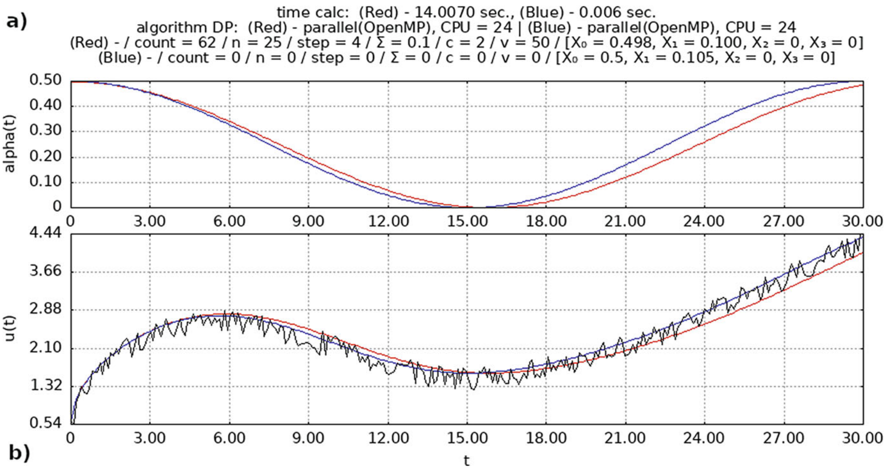

Fig. 3 Example 3. $\alpha(t_i) = X_0\cos^2(X_1 i\tau)$—periodic function. (**a**) Graphs $\alpha(t_i)$: (Blue)—"benchmark" $\vec{X}$, (Red)—restored $\vec{X}$; (**b**) Model curves: (Blue)—"benchmark", (Red)—optimization result at $\overrightarrow{X^{(0)}} = [0.1, 0.02]$, $\delta X = [0.05, 0.01]$

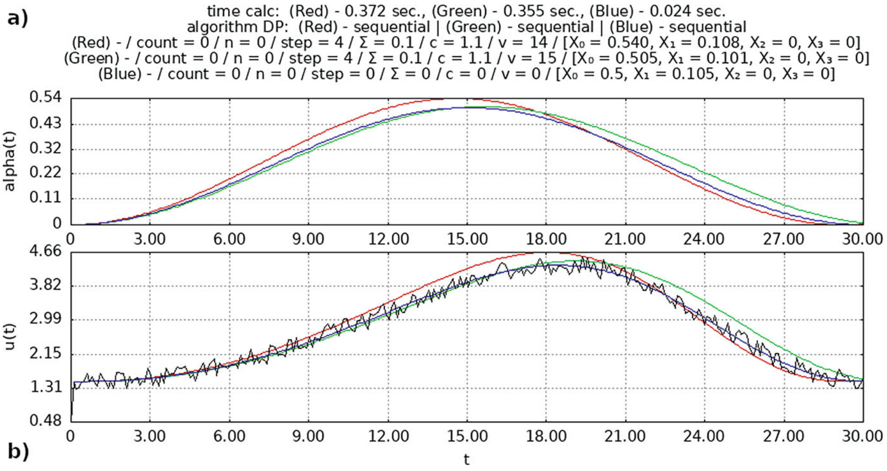

Fig. 4 Example 4. $\alpha(t_i) = X_0\sin^2(X_1 i\tau)$—periodic function. (**a**) Graphs $\alpha(t_i)$: (Blue)—"benchmark" $\vec{X}$, (Red)—restored $\vec{X}$; (**b**) Model curves: (Blue)—"benchmark", (Red)—optimization result at $\overrightarrow{X^{(0)}} = [0.005, 0.001]$, $\delta X = [0.005, 0.001]$

- Let us solve the difference inverse problem (10) by the described algorithm of the Levenberg-Marquardt method and restore the values of $\alpha(t) = \alpha\left(\vec{X}\right) = \alpha(X_0, \ldots, X_{K-1}) \in \widehat{A}$ according to $\vec{\theta}$. The (Red) and/or (Green) model curve in (Figs. 1, 2, 3, 4, and 5b).

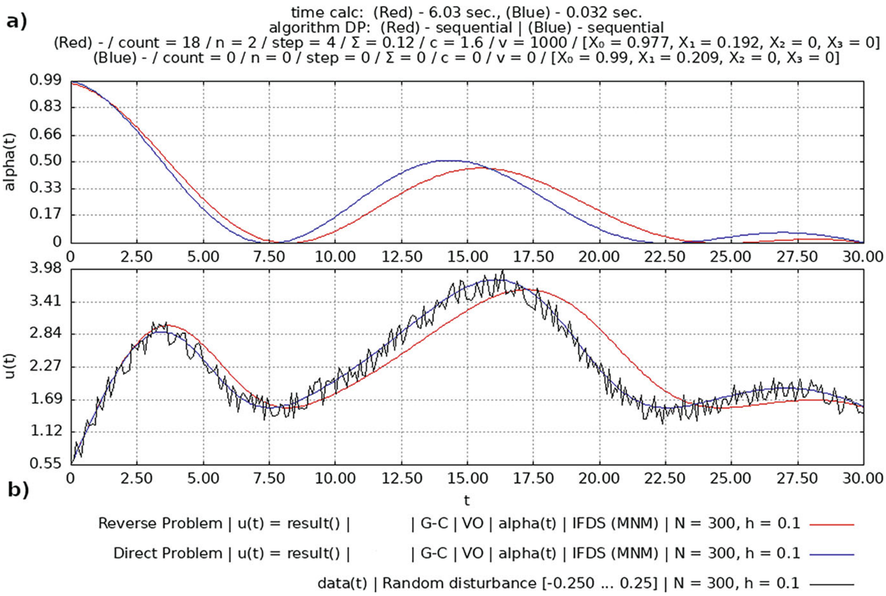

Fig. 5 Example 5. $\alpha(t_i) = (X_0((N - i)\tau/T))\cos^2(X_1 i\tau)$—linear-decreasing. (**a**) Graphs $\alpha(t_i)$: (Blue)—"benchmark" $\vec{X}$, (Red)—restored $\vec{X}$; (**b**) Model curves: (Blue)—"benchmark", (Red)—optimization result at $\overrightarrow{X^{(0)}} = [0.22, 0.0404]$, $\delta X = [0.15, 0.03]$

Remark 6 Only the most successful results are shown in (Figs. 1, 2, 3, 4, and 5). This is due to the fact that during numerical experiments on the selection of $\overrightarrow{X^{(0)}}, \delta X, c, v$ a huge amount of data was obtained to analysis the performance of the algorithm with fractional equations, many of which led to the method falling apart.

To generate $\vec{\theta}$ we will use the data of the "benchmark" solution, to which, element by element, we will add a pseudo-random disturbance in the range [−0.25,0.25] generated on the computer:

$$\theta(t_i) = u(t_i) + p(s), \quad i = 0..N-1, \quad s = \pm 0.25,$$

where, $p(s)$—is the distribution function of the discrete random variable s. This method of generating experimental data is implemented in the C programming language on the basis of rand()—function for generating pseudo-random numbers, standard library <stdlib.h>.

Calculations for solution direct and inverse problems in test examples were carried out on a personal computer purchased for work under the grant of the Russian Science Foundation № 23-71-01050 on the theme "Development of a software package for modeling and analysis of volumetric activity of radon as a precursor of strong earthquakes in Kamchatka". The personal computer has the following system characteristics: CPU—AMD Ryzen 97950X, 16 × 4.5 GHz (32 Threads); RAM—96 Gb; GPU—GeForce RTX 4090, 24 Gb, 2235 MHz, ALU 16384.

7 Conclusion

From the results we can conclude that it is possible to solve the inverse problem for an in-homogeneous fractional equation with a differentiation operator of Gerasimov-Caputo type of variable order. It is shown that with the help of mathematical unconditional optimization methods, in particular the iterative Levenberg-Marquardt method, having a mathematical model and experimental data of the process, it is possible to recover close to optimal values of α(t) the fractional derivative degree, including as a function of $\alpha(t)$ from two variables.

However, the more complex the structure of the experimental data and the type of function defining $\alpha(t)$ are, the more difficult it is to choose the parameters controlling the solution of the inverse problem of the Levenberg-Marquardt method and the starting values of $\overrightarrow{X^{(0)}}, \delta X, c, v$ at which the conditions imposed on the regularization parameter γ would be satisfied and method would converge to the optimal solution.

The continuation of this work consists in passing from the solution of inverse problems for the in-homogeneous fractional equation to inverse problems for the quadratic nonlinear fractional analogue of the Riccati equation. In turn, this will allow us to solve various inverse problems for the determination of certain parameters of the model equations of the volumetric activity of radon gas on the basis of recorded experimental data, which is of practical interest.

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