

# About Painleve property of a hydrodynamic system

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## Annotation

We represent conditions of hydrodynamic system when it passes the Painleve test. We use Kovaleskaya-Gambie method for fourth order ordinary differential system. We obtain Lorenz-like dynamic for this system.

## Introduction

The existence of magnetic fields of planets, stars and galaxy is explained by a dynamo-mechanism [1]. Mathematical aspects of dynamo effect are reduced to solutions of MHD equations. Nonlinear terms of these equations don't allow to find an analytical solutions (except private cases). Direct number simulation of 3D magnetohydrodynamics with big Reynolds number require huge computing at supercomputers [2]. So simplified MHD systems are very important to investigate for understanding main features of dynamo.

In the paper we discuss the analytical properties of a dynamical system, which is the simplest model of the dynamo. We find relation between coefficients of system in order to pass the Painleve test.

We discuss a model of process which generates the average magnetic field by a turbulence flow of viscous incompressible fluid with the alpha effect in rotating coordinate system. It is described by magnetohydrodynamics equations.

$$\begin{aligned}\partial_t \mathbf{v} + R_m (\mathbf{v} \nabla) \mathbf{v} &= Pm \Delta \mathbf{v} - \nabla p - E^{-1} Pm (\mathbf{e}_z \times \mathbf{v}) + \text{rot} \mathbf{B} \times \mathbf{B}, \\ \partial_t \mathbf{B} &= R_m \text{rot} (\mathbf{v} \times \mathbf{B}) + R_\alpha \text{rot} (\alpha \mathbf{B}) + \Delta \mathbf{B}, \\ \nabla \cdot \mathbf{v} &= 0, \\ \nabla \cdot \mathbf{B} &= 0,\end{aligned}\tag{1}$$

where  $\mathbf{v}$  – the average velocity,  $\mathbf{B}$  – the average magnetic field,  $p$  – pressure,  $\mathbf{f}$  – mass density of external forces,  $\alpha$  – tensor of the  $\alpha$ -effect,  $R_m$  – magnetic Reynolds number,  $E$  – Ekman number,  $Pm$  – Magnetic Prandtl number,  $R_\alpha$  – amplitude of the  $\alpha$ -effect,  $\mathbf{e}_z$  – the unit vector of axis of rotation.

We admit that field axially symmetric with respect to axis  $\mathbf{e}_z$ . Solenoidal fields  $\mathbf{v}$  and  $\mathbf{B}$  give the sum of the toroidal and poloidal components.

We represent the following decomposition of the velocity and magnetic field into the sum of time-dependent amplitudes and stationary poloidal (toroidal) fields products:

$$\begin{aligned}\mathbf{v} &= x_1(t) \mathbf{v}^T(\mathbf{r}) + x_2(t) \mathbf{v}^P(\mathbf{r}), \\ \mathbf{B} &= y_1(t) \mathbf{B}^T(\mathbf{r}) + y_2(t) \mathbf{B}^P(\mathbf{r}).\end{aligned}\tag{2}$$

Substitution of decomposition (2) in (1) gives system of amplitude's equations [3]:

$$\begin{aligned}
\frac{dx_1}{dt} &= R_m A_{112} x_1 x_2 + E^{-1} P_m P_{12} x_2 + F_1 + R_m L_{112} y_1 y_2 - \mu_1 x_1, \\
\frac{dx_2}{dt} &= R_m A_{211} x_1^2 + E^{-1} P_m P_{21} x_1 + F_2 + R_m L_{211} y_1^2 + R_m L_{222} y_2^2 - \mu_2 x_2, \\
\frac{dy_1}{dt} &= R_m W_{112} x_1 y_2 + R_m W_{121} x_2 y_1 + R_\alpha W_{1\alpha 2} y_2 - \eta_1 y_1, \\
\frac{dy_2}{dt} &= R_m W_{222} x_2 y_2 + R_\alpha W_{2\alpha 1} y_1 - \eta_2 y_2.
\end{aligned} \tag{3}$$

In the system we take into account that in the axially symmetric case the vector lines of any poloidal field lie in planes passing through the axis of rotation, and the lines of any toroidal field perpendicular to them. Uppercase letters denote constant coefficients. They appear after application of Galerkin's method to a system. Coefficients  $\mu_i$  and  $\eta_i$  determine the dissipation rate of velocity and magnetic fields modes from (2). In this case we assume  $P_{12} = -P_{21}$  and some other relations on coefficients  $A_{112} = -A_{211}$ ,  $L_{112} = -W_{112}$ ,  $L_{211} = -W_{121}$ ,  $L_{222} = -W_{222}$ ,  $\mu_1 = \mu_2$ ,  $\eta_1 = \eta_2$ ,  $F_1 = 0$ .

System (3) is the simplest dynamo model without a kinematic effect. In this paper we investigate its analytical properties.

## Simplified ODE system

We present the Painleve test on simplified ODE system. The Painleve test is necessary condition for the Painleve property. The formal solution is given by Laurent series near a movable singularity  $x - x_0$ . For this aim we use Kowalevski-Gambier method [4].

Main steps of Kowalevski-Gambier method are

1. substitution  $u(x) = u_0 x^p$  in order to find integer value of parameter  $p$ .
2. computation coefficients  $u(x) = u_j x^{p+j}$  for integer  $j$ . For every fixed  $j$  we have a linear algebraic system on coefficient  $u_j$  of Laurent series. If a system is consistent we have a uniformal solution for fixed  $j$ . For some values of  $j$  system can be overdefined. In this case coefficients of Laurent series are free. In order to determine  $j$  for which system is overdetermined we can calculate Fuchs indices.
3. If a linear algebra system of Laurent series coefficients for each Fuchs indices is consistent the ODE system passes Painleve test. It does not imply the Painleve property. If system is inconsistent ODE system passes test.

Let us discuss one simplified case of (3). We suppose that  $L_{222} = L_{211} = W_{121} = W_{222} = F_1 = 0$ ,  $F_2 = M$ ,  $R_\alpha W_{2\alpha 1} = R_\alpha W_{1\alpha 2} = \alpha$ ,  $P_{12} = K$ ,  $P_{21} = -K$ ,  $x_1 = u_1$ ,  $x_2 = u_2$ ,  $y_1 = u_3$ ,  $y_2 = u_4$ , independent variable  $t = x$ . So a reduced ODE system is

$$\begin{cases} \dot{u}_1 = -\lambda u_1 + K u_2 - L u_3 u_4 \\ \dot{u}_2 = -\lambda u_2 - K u_1 + M \\ \dot{u}_3 = L u_1 u_4 + \alpha u_4 - u_3 \\ \dot{u}_4 = \alpha u_3 - u_4 \end{cases} \tag{4}$$

where  $M, L, K, \lambda, \alpha$  - independent parameters.

**1 step.** We substitute  $u_i(x) = u_{i,0} x^{p_i}$ ,  $i = 1..4$  at system (4). Main terms give next system on  $p_i$ :  $p_1 - 1 = p_3 + p_4$ ,  $p_2 - 1 = p_1$ ,  $p_3 - 1 = p_1 + p_4$ ,  $p_4 - 1 = p_3$ . Solution of the linear system is  $p_1 = -2$ ,  $p_2 = -1$ ,  $p_3 = -2$ ,  $p_4 = -1$ .

We find coefficients  $u_{i,0}$  from system of main terms for calculated values of  $p_i$ ,  $i = 1..4$ :

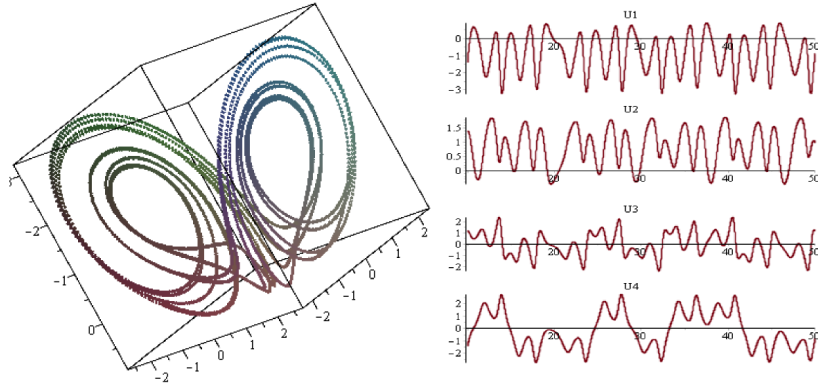


Figure 1. One velocity and two magnetic components

$$p_1 x_1 = -L x_3 x_4, p_2 x_2 = -K x_1, p_3 x_3 = L x_1 x_4, p_4 x_4 = \alpha x_3$$

From this system we get one trivial solution and four nontrivial

$$u_{1,0} = \frac{2}{L\alpha}, u_{2,0} = \frac{2K}{L\alpha}, u_{3,0} = \pm \frac{2I}{L\alpha}, u_{4,0} = \pm \frac{2I}{L},$$

where  $I$  – the imagery unit.

Let us check the first solution  $u_{3,0} = \frac{2I}{L\alpha}$ ,  $u_{4,0} = \frac{2I}{L}$ , when  $u_{3,0}$ ,  $u_{4,0}$  have the same sign. In this case Fuchs indices are irrational  $0, 1, 5/2 - 1/2\sqrt{17}, 5/2 + 1/2\sqrt{17}$ . Because of the irrational value we need to take  $u_{3,0}$ ,  $u_{4,0}$  with the different sign. For positive values of  $u_{3,0}$  and  $u_{4,0}$  the Fuchs indices are  $-1, 1, 2, 4$ . For each Fuchs index we get 3 invariants:

$$Q_1 = \frac{8(\lambda - 1)}{\alpha L}, Q_2 = -2 \frac{K^2 + \lambda^2 + 4\lambda - 4}{L\alpha}$$

$$Q_4 = ((\lambda - 2) u_{2,1} - u_{2,2}) K + 4 \frac{(\lambda - 1)(\alpha^2 \lambda - \lambda^3 - 2\lambda + 2)}{L\alpha} + \frac{\lambda(3\lambda - 2)(-u_{2,1}\lambda + M - u_{2,2})}{K}$$

Assuming  $Q_1 = Q_2 = Q_4 = 0$  we get next set of system's parameters  $\lambda = 1$ ,  $K = \pm I$ ,  $M = 0$ .

## General ODE system

In this case we assume  $P_{12} = -P_{21}$  and some other relations on coefficients  $A_{112} = -A_{211} = A$ ,  $L_{112} = -W_{112} = -L$ ,  $L_{211} = -W_{121} = -P$ ,  $L_{222} = -W_{222} = -Q$ ,  $\mu_1 = \mu_2 = \eta_1 = \eta_2 = 1$ ,  $F_1 = 0$ . For such coefficients system is more general than previous one.

$$\begin{aligned} \dot{u}_1 &= Au_1 u_2 + Ku_2 - Lu_3 u_4 - u_1 \\ \dot{u}_2 &= -Au_1^2 - Ku_1 - Pu_3^2 - Qu_4^2 + M - u_2 \\ \dot{u}_3 &= Lu_1 u_4 + Pu_2 u_3 + \alpha u_4 - u_3 \\ \dot{u}_4 &= Qu_2 u_3 + \alpha u_3 - u_4 \end{aligned} \quad (5)$$

Let us consider the Painleve test of system (5) with  $A = P$ . First we check if the pivot terms satisfy following equations

$$p_1 - 1 = p_1 + p_2, p_1 - 1 = p_3 + p_4, p_2 - 1 = 2p_1, p_2 - 1 = 2p_3, p_2 - 1 = 2p_4,$$

$$p_3 - 1 = p_1 + p_4, p_3 - 1 = p_2 + p_3, p_4 - 1 = p_2 + p_3$$

There is one case when  $p_1 = -1, p_2 = -1, p_3 = -1, p_4 = -1$ . Now we substitute expansions  $u_i = u_{0,i} x^{p_i}$  into pivot terms and get nonlinear system for coefficients  $u_{0,i}$ .

$$u_{0,1} = Iu_{0,3}, u_{0,2} = (LQu_{0,1} - P)^{-1}, u_{0,3} = \frac{IL + \sqrt{-L^2 - 4PQ}}{2Q^2}, u_{0,4} = -Qu_{0,2}u_{0,3}$$

In order to simplify  $u_{0,i}$  we put  $Q = -L^2/(4P)$ . Such simplification gives a very simple characteristic equation for the Fuchs indices  $j$  ( $j^3 - 6j^2 + 5j + 12$ ). Solving equation we obtain indices  $-1, 0, 3, 4$ . A characteristic polynomial doesn't depend on parameters  $L, P$ . There are two invariants for this system. Invariants correspond to the Fuchs values  $j = 3$  and  $j = 4$ . Detailed analysis of the invariants gives some relations on coefficients. So  $L = 2P, P = 1$ . Free parameters are  $K, \alpha$  and dependent one is  $M$ . Resulting system passes the Painleve test with  $\alpha = 3, K = 2$ .

$$\begin{aligned} \dot{u}_1 &= u_1u_2 - u_1 + 2u_2 - 2u_3u_4, \\ \dot{u}_2 &= -u_1^2 - u_2 - 2u_1 - u_3^2 + u_4^2 + \frac{1330}{99}, \\ \dot{u}_3 &= 2u_1u_4 + u_2u_3 + 3u_4 - u_3, \\ \dot{u}_4 &= -u_2u_3 + 3u_3 - u_4 \end{aligned} \tag{6}$$

If we put value of  $M$  to zero, we get a modification of (6). Direct solutions of this system gives periodic oscillations (fig. 1). If we keep constant force  $M$ , the periodic solutions will be relaxation oscillations.

## Conclusion

We have considered two cases of hydrodynamic system. For these systems we find sets of parameters to satisfy the Painleve test. It is possible to check what conservation law holds for this system.

## References

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