Bulletin KRASEC. Phys. & Math. Sci, 2014, V. 9, №. 2, pp. 56-61. ISSN 2313-0156

DOI: 10.18454/2313-0156-2014-9-2-56-61

MSC 35C05

PASSAGE THROUGH X-RAY PROTECTION HAVING THE STRUCTURE OF HOMOGENEOUS FRACTALS

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In this paper we generalize the law of Bouguer-Lambert in the case of a homogeneous fractal. With detailed analysis in terms of d-output operator generalized law of Bouguer-Lambert-Beer law, which in particular includes the classical law of optics Bouguer-Lambert-Beer.

Key words: d-operator, homogeneous fractal, fractal dimension, the law of Bouguer-Lambert-Beer

Introduction

Suppose, there is a fractal x_α in a three-dimensional Euclidean space, it has fractional dimension Euclidean space, it has fractional dimension α for which the equations $0 \leq \alpha \leq 1$ are true, along the x axis. The fractal is assumed to be isotropic and homogeneous, i.e. its dimension is constant $\alpha = const$ and does not depend on space coordinates and time. Moreover, the topological properties are assumed to be independent on space and time.

Fractal x_α points lie on the transversal axis. Axis x points, not belonging to the fractal, belong to a conjugate fractal $x_{1-\alpha}$, with the dimension $1-\alpha = const$, $0 \leq 1-\alpha \leq 1$, which is also a homogeneous fractal. In its turn, the fractal x_α is a conjugate one relative to its conjugate fractal $x_{1-\alpha}$. On x coordinate, the points of the fractal and the conjugate fractal are related as

$$x_\alpha \cup x_{1-\alpha} = x; x_\alpha \cap x_{1-\alpha} = \emptyset.$$

The second relation will be further called the property of fractal orthogonality, from which it follows that most of the physical processes occurring in the fractal and in the conjugate fractal may be independent, not affecting each other. In particular, interaction of electromagnetic radiation with the fractal and the conjugate fractal may refer to such processes.

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Definition of the problem

Damping of electromagnetic radiation beam in a continuum is described by the differential equation which describes absorption of a light beam in a medium propagating along the axis x [1]

$$dI = -kIdx.$$

Here I is the intensity of the radiation along the space coordinate ; k is the coefficient of beam attenuation in a medium.

If the continuum is composed of the combination of the fractal and the conjugate fractal, which have different physical-chemical properties, the processes of photon interaction will be different, and the corresponding differential equation describing photon beam attenuation may be written in the form of the ratio

$$dI \propto (-\tau_\alpha k_\alpha I dx_\alpha) \cup (-\tau_{1-\alpha} k_{1-\alpha} I dx_{1-\alpha}). \tag{1}$$

Here dx_α $dx_{1-\alpha}$ and are differentials on the points of the fractal x_α and the conjugate fractal which lie on the axis x; k_α is the attenuation linear coefficient in the fractal which may be represented as the sum $k_\alpha = k_{\alpha|A}(\lambda) + k_{\alpha|D}(\tau_\alpha) + k_{\alpha|S}(\tau_\alpha)$, $k_{\alpha|A}(\lambda)$, is the absorption coefficient which depends on the wave lengths of photons λ and on other factors, $k_{\alpha|D}(\tau_\alpha)$ is the inner diffraction coefficient which describes diffraction on fractal inner structures and depends on geometric and topological features of the fractal, i.e. on τ_α, $k_{\alpha|S}(\tau_\alpha)$, is the inner scattering coefficient on the fractal irregularities which also depends on τ_α ; τ_α is the topological coefficient of the fractal space for the considered process [2], $0 \leq \tau_\alpha \leq 1$; $k_{1-\alpha}$ is the attenuation linear coefficient in the conjugate fractal, which, just like in the fractal, may be presented as the sum $k_{1-\alpha} = k_{1-\alpha|A}(\lambda) + k_{1-\alpha|D}(\tau_{1-\alpha}) + k_{1-\alpha|S}(\tau_{1-\alpha})$; $k_{1-\alpha|A}(\lambda)$; $k_{1-\alpha|A}(\lambda)$ is the absorption coefficient in the conjugate fractal, $k_{1-\alpha|D}(\tau_{1-\alpha})$ is the inner diffraction coefficient in the conjugate fractal, $k_{1-\alpha|S}(\tau_{1-\alpha})$ is the inner scattering coefficient on the conjugate fractal irregularities; $\tau_{1-\alpha}$ is the topological coefficient of the conjugate fractal, $0 \leq \tau_{1-\alpha} \leq 1$.

The topological coefficients τ_α , and $\tau_{1-\alpha}$ depend on the definite topological and geometric properties of the fractal and the conjugate fractal.

When the considered process is impossible due to the topological properties, the process is topologically forbidden, then $\tau_\alpha = \tau_{1-\alpha} = 0$. The considered process is realized only through the fractals which topological coefficients are nonzero. Thus, if the topological coefficient of at least one the fractals is nonzero, the considered process is topologically allowed in this medium.

For the inner diffraction coefficients of the fractal and the conjugate fractal, the equality $k_{\alpha|D}(\tau_\alpha) = k_{1-\alpha|D}(\tau_{1-\alpha})$. must hold.

This equality holds since the fractal and the conjugate fractal share the boundary on which inner diffraction takes place. Thus, any of these two coefficients may be put into the relations, which contain them.

Strictly speaking, we may not put the sign of equality into the relation (1), since the physical dimensions of the left and the right parts are not the same in the equation during such a change

$$[dx_\alpha] = L^\alpha; [dx_{1-\alpha}] = L^{1-\alpha}; [dx] = L.$$

Due to the same reason, the term values in the right part may not be added. That is why, the sign of proportion $\propto$ is put instead of the one of equality, and the sign of union $\cup$ is put instead of the addition.

For the differentials on the fractals, $dx = dx_\alpha \cup dx_{1-\alpha}$ will be true.

Due to the orthogonality of the fractal and the conjugate fractal, the orthogonality property will be also true for the differentials

$$dx_\alpha \cap dx_{1-\alpha} = \emptyset.$$

Obtaining the solution

At first, to obtain the solution we divide the variables and put them under the integral.

$$\int \frac{dI}{I} \propto \left(-\tau_\alpha k_\alpha \int dx_\alpha\right) \cup \left(-\tau_{1-\alpha} k_{1-\alpha} \int dx_{1-\alpha}\right). \tag{2}$$

Transform the right part so that the both terms are consistent qualitatively and quantitatively between each other and are consistent with the lift part.

In order to do that, the right part should be brought to the same scale and to the same physical dimension with the left part. The first term in the right part should be multiplied by the dimension transformation coefficient $x^{1-\alpha}$, and the second term should be multiplied by the dimension transformation coefficient x^α , which brings the dimension of the right part to Euclidean space dimension where the fractal and the conjugate fractal are. Physically, it corresponds to the situation when we consider photon beam propagation along all the points on the axis x, some part of which belong to the fractal, and the rest belong to the conjugate fractal.

Then the first term in the right part should be multiplied by the fractal scale coefficient $\alpha^2\Gamma(\alpha)$, and the second term should be multiplied by scale coefficient of the conjugate fractal $(1-\alpha)^2\Gamma(1-\alpha)$. These coefficients bring the beam propagation length to the fractal effective thickness and to the conjugate fractal effective thickness $(1-\alpha)x$ which in the one-dimensional case are αx and $(1-\alpha)x$, correspondingly.

In the result, the transition from integration by the fractal of dimension α to the integral of fractional order α on the basis of d-operator is reduced to the change

$$\int dx_\alpha \to \alpha^2\Gamma(\alpha)x^{1-\alpha} \int d^\alpha x,$$

$$\int dx_{1-\alpha} \to (1-\alpha)^2\Gamma(1-\alpha)x^\alpha \int d^{1-\alpha} x.$$

Here $\int d^\alpha x$ and $\int d^{1-\alpha} x$ are the integrals of fractional orders α and $\alpha - 1$; $\Gamma(...)$ is the Euler gamma function.

After the transformations, the both terms in the right part may be added and equated with the lift part. In the result, the equation is obtained in which the variables are divided

$$\int \frac{dI}{I} = -\tau_\alpha k_\alpha \alpha^2\Gamma(\alpha)x^{1-\alpha} \int d^\alpha x - \tau_{1-\alpha} k_{1-\alpha} (1-\alpha)^2\Gamma(1-\alpha)x^\alpha \int d^{1-\alpha} x.$$

Integrating the left part, we obtain

$$\int \frac{dI}{I} = \ln(I) - \ln(C).$$

Here C is the integration constant. Integrating the right part by d-operator [3], for the first term we obtain

$$-\tau_\alpha k_\alpha \alpha^2 \Gamma(\alpha) x^{1-\alpha} \int d^\alpha x = -\frac{\tau_\alpha k_\alpha \alpha^2 \Gamma(\alpha) x^{1-\alpha}}{\alpha(\alpha)} x^\alpha + \alpha^2 \Gamma(\alpha) x^{1-\alpha} C_\alpha(x).$$

Integrating the second term, we obtain

$$-\tau_{1-\alpha} k_{1-\alpha} (1-\alpha)^2 \Gamma(1-\alpha) x^\alpha \int d^\alpha x =$$

$$= -\frac{\tau_{1-\alpha} k_{1-\alpha} (1-\alpha)^2 \Gamma(1-\alpha) x^{1-\alpha}}{(1-\alpha)(1-\alpha)} x^\alpha + (1-\alpha)^2 \Gamma(1-\alpha) x^\alpha C_{1-\alpha}(x).$$

Here $C_\alpha(x)$ and $C_{1-\alpha}(x)$ are the integration polynomials of the orders α and $1-\alpha$, which are generalization of integration constants within d-analysis [3].

De to the randomness of the second terms, they are equated to zero

$$\alpha^2 \Gamma(\alpha) x^{1-\alpha} C_\alpha(x) = 0,$$
$$(1-\alpha)^2 \Gamma(1-\alpha) x^\alpha C_{1-\alpha}(x) = 0.$$

Finally, for the sum in the right part we obtain

$$-\tau_\alpha k_\alpha \alpha^2 \Gamma(\alpha) x^{1-\alpha} \int d^\alpha x = -\tau_\alpha k_\alpha \alpha x,$$

$$-\tau_{1-\alpha} k_{1-\alpha} (1-\alpha)^2 \Gamma(1-\alpha) x^\alpha \int d^\alpha x = -\tau_{1-\alpha} k_{1-\alpha} (1-\alpha) x.$$

Equating the both parts, we obtain the solution

$$\ln(I) = -\tau_\alpha k_\alpha \alpha x - \tau_{1-\alpha} k_{1-\alpha} (1-\alpha) x + \ln(C).$$

Potentiating this expression, we obtain general explicit solution

$$I = C \exp(-\{\tau_\alpha k_\alpha \alpha + \tau_{1-\alpha} k_{1-\alpha} (1-\alpha)\} x).$$

Find a particular solution for Cauchy problem with the initial conditions x_0 and $I_0 = I(x_0)$. Expressing the integration constant in terms of the initial conditions, we obtain: $C = \exp(-\{\tau_\alpha k_\alpha \alpha + \tau_{1-\alpha} k_{1-\alpha} (1-\alpha)\} x_0 + \ln(I_0))$.

Substituting the value C into the general solution, we find the particular solution for the defined initial conditions which is the generalization of Bouguer-Lambert-Beer law for the case of light beam propagation in the medium which is the combination of the fractal and the conjugate fractal.

$$I = I_0 \exp[-\{\tau_\alpha \alpha k_\alpha + \tau_{1-\alpha} (1-\alpha) k_{1-\alpha}\} (x - x_0)]. \quad (3)$$

Here $x - x_0 \geq 0$ is the medium thickness through which the photon beam propagates, $\alpha(x - x_0)$ is the fractal effective thickness, and $(1-\alpha)(x - x_0)$ is the conjugate fractal effective thickness for which the equation $(x - x_0) = \alpha(x - x_0) + (1-\alpha)(x - x_0)$ always holds.

In particular, when $\alpha = 1$, (or $\alpha = 0$)), the effective thickness of the fractal (or the conjugate fractal) becomes equal to zero and the topological coefficient will be $\tau_\alpha = 1$ (or $\tau_{1-\alpha} = 1$)) which corresponds to the absence of the fractal (or the conjugate fractal). In these extreme cases, medium becomes continuous, and the relation (3) transforms into the classical Bouguer-Lambert-Beer law, $I = I_0 \exp[-k(x-x_0)]$ [1].

In the case, when $\tau_\alpha k_\alpha \alpha >> \tau_{1-\alpha} k_{1-\alpha}(1-\alpha)$ or $\tau_\alpha k_\alpha \alpha << \tau_{1-\alpha} k_{1-\alpha}(1-\alpha)$, beam attenuation may be neglected in the conjugate fractal or in the fractal. Then the relation will transfer to the Bouguer-Lambert-Beer law for one homogeneous fractal. Such approximation is called single-beam approximation [4].

Assume that the fractal consists of one element atoms, and the conjugate fractal consists of atoms of another element. Suppose that a beam of hard X-radiation with the wave length of less than one angstrom propagates through the medium in a second. In this case the inner diffraction may be neglected. Then attenuation coefficients for the X-radiation propagation through the fractal and the conjugate fractal will be [5]

$$k_\alpha = k_{\alpha|A}(\lambda) + k_{\alpha|S}(\tau) \approx \frac{\eta_\alpha N_A}{A_\alpha}\left(B_\alpha Z_\alpha^4 \lambda^3 + \frac{8\pi e^4 Z_\alpha}{3m^2c^4}\right),$$

$$k_{1-\alpha} = k_{1-\alpha|A}(\lambda) + k_{1-\alpha|S}(\tau) \approx \frac{\eta_{1-\alpha} N_A}{A_{1-\alpha}}\left(B_{1-\alpha} Z_{1-\alpha}^4 \lambda^3 + \frac{8\pi e^4 Z_{1-\alpha}}{3m^2c^4}\right).$$

Here $k_{\alpha|A}(\lambda)$ and $k_{1-\alpha|A}(\lambda)$ are absorption coefficients associated with the interaction of X-radiation with the atom shell inner electrons of the fractal and the conjugate fractal (inner photo effect); $k_{\alpha|S}(\tau)$ and $k_{1-\alpha|S}(\tau)$ are scattering coefficients for the fractal and the conjugate fractal material; B_α and $B_{1-\alpha}$ are semi-empirical coefficients depending on photon wave length and on atom structure of the fractal and the conjugate fractal material [3]; Z_α and $Z_{1-\alpha}$ are nuclear charges of the elements which compose the fractal and the conjugate fractal; η_α and $\eta_{1-\alpha}$ are densities of the fractal and the conjugate fractal material; N_A is the Avogadro number; A_α and $A_{1-\alpha}$ are weights of one gamma atom (atomic weight) of the fractal and the conjugate fractal material; e is the electron charge; m is the electron mass; is light velocity.

Substituting the obtained expressions for the coefficients k_α and $k_{1-\alpha}$ into (2), we obtain the Bouguer-Lambert-Beer law describing the propagation of the hard X-radiation through the medium composed of two homogeneous and orthogonal fractals

$$I = I_0 \exp\left[-\left\{\tau_\alpha \alpha \left(\frac{\eta_\alpha N_A}{A_\alpha}\left(B_\alpha Z_\alpha^4 \lambda^3 + \frac{8\pi e^4 Z_\alpha}{3m^2c^4}\right)\right) + \right.\right.$$

$$\left.\left. + \ \tau_{1-\alpha}(1-\alpha)\frac{\eta_{1-\alpha} N_A}{A_{1-\alpha}}\left(B_{1-\alpha} Z_{1-\alpha}^4 \lambda^3 + \frac{8\pi e^4 Z_{1-\alpha}}{3m^2c^4}\right)\right\}(x-x_0)\right].$$

For the problem under consideration, it is possible to accept to a high accuracy $\tau_{1-\alpha} = \tau_\alpha = 1$.

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Original article submitted: 03.12.2014