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MATHEMATICS

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FINITE GENERATED SUBGROUPS OF HYPERBOLIC GROUPS

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It is proved that finite generated subgroups of infinite index of hyperbolic groups which are not quasi Abelian are complemented with a nontrivial free factor.

Key words: hyperbolic group, free product, free product with amalgamation, set of generators, index of a subgroup, normal subgroup

A subgroup H of a group G is called freely complemented if there is a nontrivial subgroup Q in G so that a subgroup generated by the subgroups H and Q is their free product: $gp(H,Q) = H * Q$. It is clear that the subgroups of a finite group or an Abelian group can not be freely complemented. The same applies to the subgroups of a quasi-Abelian group, for example, a infinite dihedral group. The aim of this paper is to show that in a hyperbolic group which is not quasi-abelian and which is given by the representation:

$$G =< a_1, b_1, \dots, a_n,\ b_n,\ c_1, \dots, c_t,\ d_1, \dots, d_s; \tag{1}$$
$$c_1^{\gamma_1}, \dots, c_t^{\gamma_t}, [a_1,\ b_1]\ \dots\ [a_n, b_n] c_1\ \dots\ c_t, d_1\ \dots\ d_s >,$$

where $[a_i,b_i] = a_i^{-1} b_i^{-1} a_i b_i; n,s,t \geq 0; \gamma_i > 1$ – any finitely generated nontrivial subgroup of a infinite index is freely complemented.

Theorem. *Assume that G is a discrete group of orientation-preserving motions of the hyperbolic plane, G is not quasi-Abelian and G is not isomorphic to any group which has the representation:*

$$< a_1,\ b_1,\ \dots,\ a_n,\ b_n;\ ([a_1,\ b_1]\ \dots\ [a_n,\ b_n])^k\ > < a,\ b;\ a^n, b^m (ab)^k\ >,$$

where $k > 1$ and H is the finitely generated subgroup of G. Then G has a infinitely generated subgroup Q such that the subgroup generated by the subgroups H and Q is the free product of $H * Q$.

The proof is based on the following lemma of subgroups of a free product with amalgamation.

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Lemma. *Let G be a free product of two groups A and B with an amalgamated subgroup U where one of the factors is a nontrivial free product, other than the dihedral group, the subgroup U satisfies the maximum condition for subgroups and H is the finitely generated subgroup of a infinite index in G. Then G has an infinitely generated subgroup Q such that the subgroup generated by the subgroups H and Q is the free product of* $H*Q$.

Proof. Let $G = A \underset{U}{*} B$ satisfy the condition of the theorem and H is the finitely generated subgroup of a infinite index in G. Subgroup U index in group A is infinite so if H is in conjunction of the subgroup U then H has an infinite index in group A. The finitely generated subgroup of an infinite index in a free product, other than the infinite dihedral group, has the property of free complementarity (see. [1]). Moreover, the complementary factor contains a free group of rank two and therefore we can consider it to be a free group of the countable rank from the beginning.

So, we can assume that H is not in conjugation of U. Moreover, the infinite decomposition in terms of double modulus $[G\colon (H,\, U)]$ follows from the infiniteness of the index $[G\colon H]$. It follows from the theorem 1.7 in [2] that the index $[A\colon (U,\, U)]$ is infinite.

Hereafter we shall use the notations from [3].

Let S be a set of elements of the group $A \underset{U}{*} B$, and the canonical form of the element s from S has the following form

$$s = s_{1(s)} g_{2(s)} \; \cdots \; g_{n(s)}.$$

We define the set $\bar{t}_X\,(S)$, depending on S and on $X (X{=}A$ or $X = B)$ according to the next rule:

$$\bar{t}_X\,(S) = \left\{ x \in X \;\middle|\, x \equiv g_{n(s)} \bmod\, (U,\, U)\,;\, s \in S \backslash (A \cup B) \right\}.$$

For a suitable element g of G the set $\bar{t}_B\,(H^g)$ is empty and $\bar{t}_A\,(H^g) = \{U a_0 U\}$, where a_0 is any reassigned fixed element of $A \setminus U$.

We can assume that the subgroup H already has this property, that is, the element gy is the identity element and the element a_0 of $A \setminus U$ is chosen so that the subgroup R which is the subgroup generated by U and the element a_0 has an infinite index in the subgroup A.

For each element d from $D = A \cap H$ and each element t from $\bar{t}_A\,(H)$ the product td belongs to the $\bar{t}_A\,(H)$ again which equals $U a_0 U$ by assumption.

In other words, every element d of D can be represented in the form

$$d = u_1 a_0^{-1} u_2 a_0 u_3,$$

where – u_1, u_2, u_3 are suitable elements of the amalgamated subgroup U.

Therefore D is contained in R. According to the theorem 1.8 from [3], there exists an infinite subgroup Q of A that $gp(R,\, Q) = R * Q$. Now we shall show that the subgroup $\bar{H}$ which is generated by subgroups H and Q is their free product $H*Q$. In order to do this, we need to prove that the element p of H which is the product of

$$p = p_1 p_2 \; \ldots \; p_n, \tag{2}$$

where $n \geq 1$ and p_i are non-identity elements selected alternatively from the subgroups Q и H, is not identity.

If all of the elements p_i from (1), which are included into H, are simultaneously the elements of A (that is they belong to the intersection D) then the element p belongs to the free product of $Q * D$ and (2) is a normal form of the element p with respect to the decomposition of $Q * D$.

Therefore, we can assume that not all of p_i from the expansion (2) are the elements of A. Then, instead of (2) we consider other decomposition of p which can be obtained from (2) with some grouping factors p_i:

$$p = q_1 q_2 \ \dots \ q_k, \tag{3}$$

where $1 \leq k < n$,. Specifically, the element q_j is some p_i if $p_i \ \in H \setminus A$. Otherwise, q_j is the product

$$p \ = \ p_\alpha \ p_{\alpha+1} \ \cdots \ p_\beta,$$

where $1 \leq \alpha < \beta \leq n$ and all the factors p_α, $p_{\alpha\ +\ 1}$, ..., p_β belong to the factor A but the element $p_{\alpha-1}$ (in the case when $\alpha > 1$) and the element $p_{\beta+1}$ (in the case when $\beta < n$) do not belong to A.

Thus, in the expansion (3) the factors are selected by turns from $H \setminus A$ and the free product of $Q * R$. In this case, if q_j is the element of $Q * R$ then it is not included into the free factor R. It means that for each element r_1, r_2 from R the product of $r_1 q_{j_0} r_2$ does not belong to the subgroup R.

On the other hand, if $h \ \in H \setminus A$ and $h = h_1 h_2 \ ... \ h_m$ is its canonical form then from $\bar{t}_A\,(H) \subseteq R$ we have that $h_m \ \in A$ implies $h_m \ \in R$ (and $h_1 \ \in A$ implies $h_1 \ \in R$)).

Hence, the canonical form of the element p has at least k syllables. But this means that the element p is not equal to unity and, thus, the subgroup H_2 is the free product of $H * Q$.

The lemma is proved.

We proceed now to the proof of the theorem.

If in the group with the representation (1) the parameter $s \ >$ then G is a free product of cyclic groups of the second order (and the infinite dihedral group is quasi-Abelian).

If $s \ > \ = 0$ then the representation (1) becomes a representation of the form

$$G = < a_1,\ b_1,\ \dots,\ a_n,\ b_n,\ c_1,\ \dots,\ c_t; c_1^{\gamma_1}, \dots,\ c_t^{\gamma_t}, [a_1,\ b_1]\ \dots\ [a_n,\ b_n] c_1 \dots_t >,$$

Now, if $n > 0$ and $t > 1$ then the group G can be represented as follows

$$G = gp(a_1,\ b_1,\ \dots,\ a_n,\ b_n) \underset{U}{*} gp(c_1,\ \dots,\ c_t),$$

where $U = gp([a_1,\ b_1]\ ...\ [a_n,\ b_n]) = gp(c_t^{-1} c_{t-1}^{-1} ... c_1^{-1})$.

The same is in the cases when $n > 1$ и $t = 0$ and when $n = 0$, $t > 3$.

In the latter case it may turn out that G is a free product of two dihedral groups with an amalgamated cyclic subgroup:

$$G = < c_1,\ c_2,\ c_3,\ c_4,\ c_1^2, c_2^2, c_3^2, c_4^2,\ c_1 c_2 c_3 c_4 > .$$

Then the group G is quasi-Abelian again. Generated by the element $c_1 c_2 (c_{13})^2$ the cyclic subgroup has the finite index in G indeed. If $s = n = 0$, $t \leq 2$, then the group G is finite. Thus, all hyperbolic groups, except for the cases with $s = 0, n > 0, t = 1$ or $s = 0, n = 0, t = 3$ in the representation (1), satisfy the conditions of the lemma. The theorem is proved. □

We should note that in the paper [4], the necessary and sufficient conditions for the existence of free complementarity for quasi-convex subgroups of an infinite index in a hyperbolic group were obtained by other methods.

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