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ABOUT A METHOD OF RESEARCH OF THE NON-LOCAL PROBLEM FOR THE LOADED MIXED TYPE EQUATION IN DOUBLE-CONNECTED DOMAIN

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In the present paper an existence and uniqueness of solution of the non-local boundary value problem for the loaded elliptic-hyperbolic type equation on the third order in double-connected domain was investigated. The uniqueness of solution was proved by the extremum principle for the mixed type equations, and existence was proved by the method of integral equations.

Key words: loaded equation, elliptic-hyperbolic type, double-connected domain, an extremum principle, existence of solution, uniqueness of solution, method of integral equations

Introduction

We shall notice that with intensive research on problem of optimal control of the agroeconomical system, regulating the level of ground waters and soil moisture, it has become necessary to investigate a new class of equations called "LOADED EQUATIONS". Such equations were investigated in first in the works by N.N. Nazarov and N.Kochin, but they didn't use the term "LOADED EQUATIONS". For the first time, the most general definition of the LOADED EQUATIONS was given and various loaded equations were classified in detail by A.M. Nakhushev [1].

Let's notice that non-local problems for the loaded elliptic-hyperbolic type equations in double-connected domains have not been investigated. In the given paper, uniqueness of solution of the non-local boundary value problem for the loaded elliptic-hyperbolic type equation in double-connected domain was proved, and the method of the solvability of the investigated problem was presented.

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The statement of the Problem

Let's consider the equation

$$u_{xx}+sgn(xy)u_{yy}+\Re(x,y)=0 \tag{1}$$

in double-connected domain of Ω, bounded with to(two) lines:

$$\sigma_1:\ x^2+y^2=1\ ;\quad \sigma_2:\ x^2+y^2=q^2;\ at x>0,y>0;$$

$$\sigma_1^*:\ x^2+y^2=1\ ;\quad \sigma_2^*:\ x^2+y^2=q^2,\ at x<0,y<0;$$

and characteristics:

$$A_jA_j^*:x-y=(-1)^{j+1};\quad B_jB_j^*:x-y=(-1)^{j-1}\cdot q,\ \ (j=1,2)$$

at $x\cdot y<0$, of the equations (1), where $A_1(1;0)$, $A_2(0;1)$, $A_1^*(0;-1)$, $A_2^*(-1;0)$, $B_1(q;0)$, $B_2(0;q)$, $B_1^*(0;-q)$, $B_2^*(-q;0)$, $0<q<1$, $(j\ =\ 1,2)$.

$$\Re(x,y)=\frac{1-sgn(xy)}{2}\cdot[\lambda_1\cdot\Re_1(x,y)+\lambda_2\cdot\Re_2(x,y)],\quad \lambda_1,\lambda_2=const>0;$$

$$\Re_1(x,y)=\frac{1+sgn(xy+x^2)}{2}u(x,0),\quad \Re_2(x,y)=\frac{1+sgn(xy+y^2)}{2}u(0,y).$$

Let's enter designations:

$$E_j\left(\frac{q-(-1)^j}{2};\frac{q+(-1)^j}{2}\right),E_j^*\left(\frac{(-1)^{j-1}-q}{2};\frac{(-1)^j-q}{2}\right),$$

$$C_j\left((-1)^{j-1}\frac{q}{2};(-1)^j\frac{q}{2}\right),\ \Omega_0=\Omega\cap(x>0)\cap(y>0),\Omega_0^*=\Omega\cap(x<0)\cap(y<0),$$

$$\Delta_1=\Omega\cap(x+y>q)\cap(y<0);\Delta_1^*=\Omega\cap(x+y<-q)\cap(y<0),$$

$$\Delta_2=\Omega\cap(x+y>q)\cap(x<0),\ \Delta_2^*=\Omega\cap(x+y<-q)\cap(y>0),$$

$$D_1=\Omega\cap(-q<x+y<q)\cap(y<0),\quad D_2=\Omega\cap(-q<x+y<q)\cap(y>0),$$

$$D_0=\Omega_0\cup\Delta_1\cup\Delta_2,\quad D_0^*=\Omega_0^*\cup\Delta_1^*\cup\Delta_2^*,$$

$$I_j=\{t:\ 0<t<q\};\quad I_{2+j}=\left\{t:0<(-1)^{j-1}t<1\right\},$$

where $t=\begin{cases} x\ at\ \ j=1,\\ y\ at\ \ j=2.\end{cases}$

Let's designate, through $\theta_1\left(\frac{x+1}{2};\frac{x-1}{2}\right)$, $\theta_2\left(\frac{y-1}{2};\frac{y+1}{2}\right)$ points of intersections of characteristics of the equation (1) with leaving points $(x,0)\in A_1B_1$ and $(0,y)\in A_2B_2$ with characteristics $A_1A_1^*$ and $A_2A_2^*$, accordingly.

In the domain of Ω the following problem is investigated

Task. ***(Problem A.)*** *To find function $u(x,y)$ with following properties:*
1) $u(x,y)\in C(\bar{\Omega})$;
2) $u(x,y)$ is the regular solution of the equation (1) in the domain of $\Omega\backslash(xy=0)\backslash(x+y=\pm q)$, besides, $u_y\in C(A_1B_1\cup A_2^*B_2^*)$, $u_x\in C(A_2B_2\cup A_1^*B_1^*)$ at that $u_x(0,t)$, $u_y(t,0)$ can tend infinity of an order of less unit at $t\to\pm q$, and finite at $t\to\pm 1$;
3) $u(x,y)$ satisfies gluing conditions on lines of changing type:

$$u_y(x,-0)=u_y(x,+0),\ \textit{in regular intervals on}\,(x,0)\in A_1B_1\cup A_2^*B_2^*,$$

$$u_x(-0,y)=u_x(+0,y),\ \textit{in regular intervals on}\ (0,y)\in A_1^*B_1^*\cup A_2B_2,$$

4) $u(x,y)$ satisfies to boundary conditions:

$$u(x,y)\left|_{\sigma_j}=\phi_j(x,y);\ \ (x,y)\in\overline{\sigma_j},\right. \tag{2}$$

$$u(x,y)\Big|_{\sigma_j^*}=\phi_j^*(x,y);\ \ (x,y)\in\overline{\sigma_j^*}, \tag{3}$$

$$\frac{d}{dx}u\left(\theta_1(x)\right)=a_1(x)u_y(x;0)+b_1(x),\quad q<x<1, \tag{4}$$

$$\frac{d}{dy}u\left(\theta_2(y)\right)=a_2(y)u_x(0;y)+b_2(y),\quad q<y<1, \tag{5}$$

$$u(x,y)\Big|_{B_jB_j^*}=g_j(t),\ \ t\in\overline{I_j}, \tag{6}$$

where $\phi_j(x,y)$, $\phi_j^*(x,y)$, $g_j(t), a_j(t), b_j(t)$, $(j=1,2)$ is given function, at that:

$$\left.\begin{array}{l} g_j\left(\frac{q}{2}\right)=g_j^*\left(\frac{q}{2}\right),\ \ \phi_2(q,0)=g_1(q),\ \phi_2(0,q)=g_2(q),\\ \phi_2^*(0,-q)=g_1(-q),\ \phi_2^*(-q,0)=g_2(-q),\end{array}\right\} \tag{7}$$

$$\phi_j(x,y)=(xy)^{\gamma}\overline{\phi_j}(x,y);\ \overline{\phi_j}(x,y)\in C\left(\overline{\sigma_j}\right),\ 2<\gamma<3 \tag{8}$$

$$\phi_j^*(x,y)=(xy)^{\gamma}\overline{\phi_j^*}(x,y);\ \overline{\phi_j^*}(x,y)\in C\left(\overline{\sigma_j^*}\right),\ 2<\gamma<3, \tag{9}$$

$$g_j(t)\in C\left(\overline{I_j}\right)\cap C^2\left(I_j\right), \tag{10}$$

$$a_j(t),\ b_j(t)\in C\left[q,1\right]\cap C^2\left(p,q\right); \tag{11}$$

Theorem. *If conditions (7) - 11) and*

$$a_1(x)>\frac{1}{2},\quad a_2(y)>\frac{1}{2}\quad; \tag{12}$$

are satisfied, then the solution of the ***Problem A*** *exists and it is unique.*

Proof. Note, that the solution the equation (1) in hyperbolic domains looks like:

$$u(x,y)=f_1(x+y)+f_2(y-x)+\lambda_2\int\limits_q^y(y-t)z(0,t)dt, at y>0,\ x<0; \tag{13}$$

$$u(x,y)=f_1(x+y)+f_2(x-y)+\lambda\int\limits_y^{-q}(t-y)z(0,t)dt, at x>0,\ y<0; \tag{14}$$

Owing to (13) we will receive, that the solution of Cauchy problem of the equation (1) satisfies conditions $u(0,y)=\tau_2(y)$ and $u_x(0,y)=\nu_2(y)$ in the domain of Δ_2, looks like:

$$u(x,y)=\frac{\tau_2(x+y)+\tau_2(y-x)}{2}+\frac{1}{2}\int\limits_{y-x}^{y+x}\nu_2(t)dt-\frac{\lambda}{2}\int\limits_q^{y+x}(x+y-t)\tau_2(t)dt-$$

$$-\frac{\lambda}{2}\int\limits_q^{y-x}(y-x-t)\tau_2(t)dt+\lambda\int\limits_q^y(y-t)\tau_2(t)dt. \tag{15}$$

From here, by virtue, (5) we will receive a functional relation between $\tau_2(y)$ and $\nu_2(y)$ from the domain of Δ_2 on the piece A_2B_2:

$$\frac{1}{2}\tau_2'(y)+\frac{1}{2}\nu_2(y)-\lambda_2\int\limits_q^y\tau_2(t)dt+\frac{\lambda_2}{2}\int\limits_q^{\frac{y+1}{2}}\tau_2(t)dt=\nu_2(y)a_2(y)+b_2(y),$$

i.e.,

$$(2a_2(y)-1)\,\nu_2(y)=\tau_2'(y)+\lambda_2\int\limits_q^{\frac{y+1}{2}}\tau_2(t)dt+2\lambda_2\int\limits_q^y\tau_2(t)dt-b_2(y). \tag{16}$$

Confirm, that a functional relation between $\tau_1(x)$ and $\nu_1(x)$we will obtain by the same method, from the solution of Cauchy problem for the equation (1) satisfying to conditions $u(x,0)=\tau_1(x)$; $u_y(x,0)=\nu_1(x)$ in the domain of Δ_1, and with the account of conditions (4), which presented on the form:

$$(2a_1(x)-1)\,\nu_1(x)=\tau_1'(x)+\lambda_1\int\limits_q^{\frac{x+1}{2}}\tau_1(t)dt+2\lambda_1\int\limits_q^y\tau_1(t)dt-b_1(x). \tag{17}$$

□

The uniqueness of solution of the Problem A

It is known that, if the homogeneous problem has only trivial solution than a solution of the accordingly non-uniform problem is unique, therefore, we need to prove that the homogeneous problem has only trivial solution.

Let $b_1(y) \equiv b_2(x) \equiv 0$ then, from (16) and (17), accordingly, we will receive:

$$(2a_2(y)-1)\,\nu_2(y) = \tau_2'(y) + \lambda_2 \int_q^{\frac{y+1}{2}} \tau_2(t)dt + 2\lambda_2 \int_q^{y} \tau_2(t)dt, \quad (18)$$

and

$$(2a_1(x)-1)\,\nu_1(x) = \tau_1'(x) + \lambda_1 \int_q^{\frac{x+1}{2}} \tau_1(t)dt + 2\lambda_1 \int_q^{x} \tau_1(t)dt. \quad (19)$$

Lemma 1. *If $b_1(y) \equiv b_2(x) \equiv 0$ and satisfied conditions (12) that solution $u(x,y)$ of the equation (1) reaches the positive maximum, and the negative minimum in closed domain of $\overline{D_0}$ only on $\overline{\sigma_1}$ and $\overline{\sigma_2}$.*

Proof. According to the extremum principle for the hyperbolic [2] and elliptic equations [3], the solution $u(x,y)$ of the equation (1) can reach the positive maximum and the negative minimum in closed domain of $\overline{D_0}$ only on $A_2B_2 \cup A_1B_1$ and $\overline{\sigma_1} \cup \overline{\sigma_2}$. We need to prove, that the solution $u(x,y)$ of the equation (1) can't reach the positive maximum and the negative minimum in $A_2B_2 \cup A_1B_1$.

Let the function $u(x,y)$ $(u(0,y) = \tau_2(y))$ reach the positive maximum (the negative minimum) on some point of$y_0 \in A_2B_2$, then from (18) and based on (12) we will receive

$$\nu_2(y_0) = \frac{\lambda_2}{2a_2(y_0)-1}\left[\int_q^{\frac{y_0+1}{2}} \tau_2(t)dt + 2\int_q^{y_0} \tau_2(t)dt\right] > 0 \quad (\nu_2(y_0) < 0).$$

Therefore, in the view of to the gluing condition, we have $\nu_2^+(y_0) > 0 \quad \left(\nu_2^+(y_0) < 0\right)$, and it contradicts the known Zareba-Zero principle [3], according to which in a point of a positive maximum (a negative minimum) should be $\nu_2(y_0) < 0 \quad (\nu_2(y_0) > 0)$. Thus, $u(x,y)$ does not reach the positive maximum (the negative minimum) on the point of $y_0 \in A_2B_2$.

Owing to (19) and (12), it is similarly proved that the function $u(x,y)$ does not reach the positive maximum and the negative minimum in the interval of A_1B_1.

The Lemma 1 is proved. □

As $\tau_2(y)$ does not reach the positive maximum and the negative minimum in the interval of A_2B_2 then, $\tau_2(y) = const$ in the interval of A_2B_2. Further by virtue $u(x,y) \in C(\bar{\Omega})$, we will receive $\tau_2(1) = \phi_1(0,1) \equiv 0$, $\tau_2(q) = \phi_2(0,q) \equiv 0$ at the $\phi_j(x,y) \equiv 0$. From here, as $\tau_2(y) \in C\left(\overline{A_2B_2}\right)$ we will conclude that, $\tau_2(y) \equiv 0$ in $\overline{A_2B_2}$. Therefore, if $b_2(y) \equiv 0$ and $a_2(y) > \frac{1}{2}$, then from (18) we will get $\nu_2(y) \equiv 0$. Also, by the same method, we can get $\tau_1(x) \equiv 0$ in $\overline{A_1B_1}$ and $\nu_1(x) \equiv 0$.

Hence, owing to uniqueness of solution of the Cauchy problem for the equation (1), we will have $u(x,y) \equiv 0$ in domains $\overline{\Delta_1}$ and $\overline{\Delta_2}$.

Consequently, in the view of the **Lemma 1**, at $\phi_1(x,y) \equiv \phi_2(x,y) \equiv 0$ we will deduce, that $u(x,y) \equiv 0$ in closed domain of $\overline{\Omega_0}$. Thus, $u(x,y) \equiv 0$ in $\overline{D_0}$.

Further, owing to condition 7 and considering the uniqueness of solution of the Gaursat problem at$g_j(t) \equiv 0$, $t \in \overline{I_j}$ we will receive $u(x,y) \equiv 0$ in $\overline{D_j}$ (j = 1,2).

Takes place:

Lemma 2. *The solution $u(x,y)$ of the equation (1) reach the positive maximum and the negative minimum in closed domain $\overline{D_0^*}$ only on $A_2^*B_2^* \cup A_1^*B_1^*$, $\overline{\sigma_1^*}$ and $\overline{\sigma_2^*}$.* (***Lemma 2 will be proved similarly as Lemma 1***)

On the basis of **Lemma 2**, on account of 3 at $\phi_j^*(x,y) \equiv 0$, we will receive $u(x,y) \equiv 0$ in the domain of $\overline{D_0^*}$. Thus, the solution of homogeneous **Problem A** is identically equal to zero in the domain of Ω. (***The uniqueness of solution of the*** **Problem A** ***is proved*.)**

The existence of solution of the problem I

It's known, that the solution $u(x,y)$ of the Nyman problem (**Problem N**) [2] satisfying conditions:

1) $u(x,y) \in C(\overline{D_0}) \cap C^2(D_0 \backslash xy = 0) \cap C^1(D_0)$ is the solution of the equation (1);
2) $u_y \in C(A_1B_1)$, $u_x \in C(A_2B_2)$, at that $u_x(0,t), u_y(t,0)$ can tend to infinity of an order of less unit at $t \to q$, and finite at $t \to 1$;
3) satisfies to boundary conditions (2) (j=1,2) and $u_x(0,y) = \nu_2(y)$, $u_y(x,0) = \nu_1(x)$ is exists and is unique and represented in the form [2]

$$u(x,y) = \int\limits_{\sigma_1} \phi_1(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,y)dS - \int\limits_{\sigma_2} \phi_2(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,y)dS +$$

$$+\int\limits_q^1 \nu_1^+(t)G(t,0;x,y)dt + \int\limits_q^1 \nu_2^+(t)G(0,t;x,y)dt, \quad (20)$$

where $G(\xi,\eta;x,y)$ is Green's function of the problem N for the Laplace equation in the domain of Ω_0, which looks like:

$$G(\xi,\eta;x,y) = \frac{1}{2\pi}\ln\left|\frac{\theta_1\left(\frac{\ln\nu+\ln\overline{\mu}}{2\pi ir}\right)\theta_1\left(\frac{\ln\overline{\nu}+\ln\overline{\mu}}{2\pi ir}\right)\theta_1\left(\frac{\ln(-\nu)+\ln\overline{\mu}}{2\pi ir}\right)\theta_1\left(\frac{\ln(-\overline{\nu})+\ln\overline{\mu}}{2\pi ir}\right)}{\theta_1\left(\frac{\ln\nu-\ln\mu}{2\pi ir}\right)\theta_1\left(\frac{\ln\overline{\nu}-\ln\mu}{2\pi ir}\right)\theta_1\left(\frac{\ln(-\nu)-\ln\mu}{2\pi ir}\right)\theta_1\left(\frac{\ln(-\overline{\nu})-\ln\mu}{2\pi ir}\right)}\right|, \quad (21)$$

where $\nu = \xi + i\eta$, $\overline{\nu} = \xi - i\eta$, $\mu = x+iy$, $\overline{\mu} = x-iy$, $r = \frac{1}{\pi i}\ln q$, $i^2 = -1$, $\theta_1(\xi) = \theta_1\left(\xi \,|-\frac{1}{r}\right)$ is theta function.

From (20) $y = 0$ and$x = 0$, we will get functional relations between $\tau_2^+(y)$, $\nu_2^+(y)$ and $\tau_1^+(x)$, $\nu_1^+(x)$, respectively, on the pieces A_2B_2 and A_1B_1 getting from the domain Ω_0:

$$\tau_1^+(x) = \sum_{k=1}^{2}(-1)^{k-1}\int\limits_{\sigma_k}\phi_k(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,0)dS + \int\limits_q^1 \nu_1^+(t)G(t,0;x,0)dt +$$

$$+\int\limits_q^1 \nu_2^+(t)G(0,t;x,0)dt. \quad (22)$$

$$\tau_2^+(y) = \sum_{k=1}^{2}(-1)^{k-1}\int\limits_{\sigma_k}\phi_k(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;0,y)dS + \int\limits_q^1 \nu_1^+(t)G(t,0;0,y)dt +$$

$$+\int_q^1 \nu_2^+(t)G(0,t;0,y)dt, \tag{23}$$

After differentiating equalities (22) by x and (23) by y, we obtain

$$\tau_1'^+(x) = \int_q^1 \nu_1^+(t)\frac{\partial G(t,0;x,0)}{\partial x}dt + \int_q^1 \nu_2^+(t)\frac{\partial G(0,t;x,0)}{\partial x}dt + F_1'(x), \tag{24}$$

where $F_1(x) = \sum\limits_{k=1}^{2}(-1)^{k-1}\int\limits_{\sigma_k}\varphi_k(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;x,0)dS$, and

$$\tau_2'^+(y) = \int_q^1 \nu_1^+(t)\frac{\partial G(t,0;0,y)}{\partial y}dt + \int_q^1 \nu_2^+(t)\frac{\partial G(0,t;0,y)}{\partial y}dt + F_2'(y), \tag{25}$$

where $F_2(y) = \sum\limits_{k=1}^{2}(-1)^{k-1}\int\limits_{\sigma_k}\phi_k(\xi,\eta)\frac{\partial}{\partial n}G(\xi,\eta;0,y)dS.$

Further, the equations (16) and (17) we will rewrite in the form:

$$\tau_j'(t) + \lambda_j\int_q^{\frac{t+1}{2}} K(t,z)\tau_j'(z)dz = \tilde{F}_j(t), \tag{26}$$

where

$$\tilde{F}_j(t) = (2a_j(t)-1)\nu_j(t) + b_j(t) + \lambda_j\tau_j(q)\left(\frac{5t+1}{2} - 3q\right), \tag{27}$$

$$K(t,z) = \begin{cases} \frac{5y+1}{2} - 3z, & q \le z \le t; \\ \frac{t+1}{2} - z, & t \le z \le \frac{t+1}{2}, \end{cases} \qquad t = \begin{cases} x \; at \; j=1, \\ y \; at \; j=2. \end{cases} \tag{28}$$

Note that we will search the function $\nu_j(t)$ from the class of $C^2(q,1)$, which can tend to infinity of an order of less unit at $t \to q$, and finite at $t \to 1$. From here and owing to account (11), we will decide $\tilde{F}_j(t) \in C^2(q,1)$ and that $\tilde{F}_j(t)$ can tend to infinity of an order of less unit at $t \to q$, and finite at $t \to 1$.

Hence, by virtue

$$|K_j(t,z)| \le const,$$

we will conclude that the equations (26) are the Volterra integral equations of the second kind which unequivocally solved by the method of consecutive approach [2] and its solution is represented in the form:

$$\tau_j'(t) = \lambda_j\int_q^{\frac{t+1}{2}} R_j(t,z,\lambda_j)\tilde{F}_j(z)dz + \tilde{F}_j(t), \tag{29}$$

where $R_j(t,z,\lambda_j)$ are resolves of the kernels $K_j(t,z)$, and that

$$|R_j(t,z,\lambda_j)| \le const, (j=1,2). \tag{30}$$

Further, having excluded $\tau_j'(t)$ from the relations (24) and (29), we will receive system of the integral equations

$$\begin{cases} (2a_1(x)-1)\nu_1(x)+\lambda_1\int\limits_q^{\frac{x+1}{2}}\tilde{R}_1(x,z,\lambda_1)\nu_1(z)dz-\int\limits_q^1\nu_1^+(t)\frac{\partial G(t,0;x,0)}{\partial x}dt=\Phi_1(x)\\ (2a_2(y)-1)\nu_2(y)+\lambda_2\int\limits_q^{\frac{y+1}{2}}\tilde{R}_2(y,z,\lambda_2)\nu_2(z)dz-\int\limits_q^1\nu_2^+(t)\frac{\partial G(0,t;0,y)}{\partial y}dt=\Phi_2(y)\end{cases} \quad (31)$$

where

$$\Phi_1(x)=\int\limits_q^1\nu_2^+(t)\frac{\partial G(0,t;x,0)}{\partial x}dt-\lambda_1\int\limits_q^{\frac{x+1}{2}}R_1(x,z,\lambda_1)\tilde{b}_1(z)dz-\tilde{b}_1(x)+F_1'(x), \quad (32)$$

$$\tilde{b}_1(x)=b_1(x)+\frac{\lambda_1\phi_2(q,0)}{2}(5x-6q+1);$$

$$\Phi_2(y)=\int\limits_q^1\nu_1^+(t)\frac{\partial G(t,0;0,y)}{\partial y}dt-\lambda_2\int\limits_q^{\frac{y+1}{2}}R_2(y,z,\lambda_2)\tilde{b}_2(z)dz-\tilde{b}_2(y)+F_2'(y), \quad (33)$$

$$\tilde{b}_2(y)=b_2(y)+\frac{\lambda_2\phi_2(0,q)}{2}(5y-6q+1).$$

$$G(t,0;x,0)=G(0,t;0,x)=K_1(x,t),$$

$$K_1(x,t)=\frac{1}{\pi}\left[\frac{\ln t\ln x}{\ln q}+K_1^*(x,t)+K_1^*(x,-t)\right] \quad (34)$$

$$G(t,0;0,x)=G(0,t;x,0)=K_2^(x,t)=K_1^(x,-it), \quad (35)$$

$$K_1^*(x,t)=\ln\left|\frac{1-tx}{t-x}\right|+\sum_{n=1}^{\infty}\left(\ln\left|\frac{1-q^{2n}tx}{t-q^{2n}x}\right|+\ln\left|\frac{1-q^{-2n}tx}{t-q^{-2n}x}\right|\right). \quad (36)$$

Further, after some simplifications, from (31)-(36) we can get the system of singularity integral equation concerning $\nu_1(x)$ and $\nu_2(y)$ which represented on(in) the form [2]

$$\begin{cases} \nu_1^(x)=\int\limits_q^1\nu_1(t)A_1(x,t)dt+\tilde{\Phi}_1(x)\\ \nu_2^(x)=\int\limits_q^1\nu_2(t)A_2(x,t)dt+\tilde{\Phi}_2(x)\end{cases}; \quad (37)$$

where $\tilde{\Phi}_j(x)=\frac{\Phi_j(x)}{2a_j(x)-1}$;

$$A_j(x,t)=\begin{cases}\lambda_j\frac{\tilde{R}_j(x,z,\lambda_j)}{2a_j(x)-1}-\frac{\bar{K}_1(x,t)}{2a_j(x)-1}, & q\le z\le\frac{x+1}{2};\\ \frac{\bar{K}_1(x,z)}{2a_j(x)-1}, & \frac{x+1}{2}\le z\le 1,\end{cases} \quad (38)$$

$$\bar{K}_1(x,t)=\frac{2\ln|t|}{x\ln q}+K(x,t)+K(x,-t), \quad (39)$$

$$K(x,t)=\frac{1}{\pi}\left[\frac{1}{t-x}-\frac{t}{1-tx}+\sum_{n=1}^{\infty}\left(\frac{q^{2n}}{t-q^{2n}x}-\frac{q^{2n}t}{1-q^{2n}tx}-\frac{q^{-2n}t}{1-q^{-2n}tx}+\frac{q^{-2n}}{t-q^{2n}x}\right)\right]. \quad (40)$$

Further, we will investigate the function $\Phi_1(x)$:

$$\Phi_1(x)=\int_q^1 \nu_2^+(t)\bar{K}_1(x,it)dt-\lambda_1\int_q^{\frac{x+1}{2}} R_1(x,z,\lambda_1)\tilde{b}_1(z)dz-\tilde{b}_1(x)+$$

$$+\int_{\sigma_1}\phi_1(\xi,\eta)\frac{\partial}{\partial x}\left(\frac{\partial G(\xi,\eta;x,0)}{\partial\xi}\right)d\eta-\int_{\sigma_1}\phi_1(\xi,\eta)\frac{\partial}{\partial x}\left(\frac{\partial G(\xi,\eta;x,0)}{\partial\eta}\right)d\xi+$$

$$-\int_{\sigma_2}\phi_2(\xi,\eta)\frac{\partial}{\partial x}\left(\frac{\partial G(\xi,\eta;x,0)}{\partial\xi}\right)d\eta+\int_{\sigma_2}\phi_2(\xi,\eta)\frac{\partial}{\partial x}\left(\frac{\partial G(\xi,\eta;x,0)}{\partial\eta}\right)d\xi=$$

$$=\int_q^1 \nu_2^+(t)\bar{K}_1(x,it)dt-2\int_0^1\phi_1(\xi,\eta)\frac{\xi}{\sqrt{1-\xi^2}}\frac{\partial}{\partial x}\left(\frac{\partial G(\xi,\eta;x,0)}{\partial\xi}\right)d\xi-$$

$$-\int_0^1\phi_1(\xi,\eta)\frac{\partial}{\partial\xi}\left(\frac{\partial G(\xi,\eta;x,0)}{\partial\eta}\right)d\xi+2\int_0^q\phi_2(\xi,\eta)\frac{\xi}{\sqrt{q^2-\xi^2}}\frac{\partial}{\partial x}\left(\frac{\partial G(\xi,\eta;x,0)}{\partial\xi}\right)d\xi+$$

$$+\int_0^q\phi_2(\xi,\eta)\frac{\partial}{\partial x}\left(\frac{\partial G(\xi,\eta;x,0)}{\partial\eta}\right)d\xi+\beta(x)=$$

$$=\alpha_1(x)+\alpha_2(x)+\alpha_3(x)+\alpha_4(x)+\alpha_5(x)+\beta(x)$$

where $\beta(x)=-\lambda_1\int_q^{\frac{x+1}{2}} R_1(x,z,\lambda_1)\tilde{b}_1(z)dz-\tilde{b}_1(x)$.

Owing to account (9), (11), (21), (30), (39) and (40), we will get (***see*** [2]):

$$|\beta(x)|\le const,\quad |\alpha_1(x)|\le const,\quad |\alpha_2(x)|\le const,\quad |\alpha_3(x)|\le const,$$

$$|\alpha_4(x)|\le (x-q)^{\gamma-3}const,\quad |\alpha_5(x)|\le (x-q)^{\gamma-3}const,\quad (2<\gamma<3). \quad (41)$$

Hence, by virtue (12) and (41), we will conclude that $\tilde{\Phi}_1(x)\in C^2(q,1)$, and $\tilde{\Phi}_1(x)$ can tend to infinity an order of less one at $x\to q$, and at $x\to 1$ it is limited.

It is similarly proved, that $\tilde{\Phi}_2(y)\in C^2(q,1)$, and $\Phi_2(y)$ can tend to infinity an order of less one at $y\to q$, and at $y\to 1$ it is limited.

Thus, the system of integral equation (37) is reduced to the Fredholm integral equations of the second kind, by the known method Karleman-Vekua [4], just as in works [2], [3].

Note, that unique solvability of the Fredholm integral equations of the second kind follows from the uniqueness of solution of the **Problem A** and from the theory integral equations.

Solving the system of integral equations (37), we will found $\nu_1(x)$ and $\nu_2(y)$ [2], further owing to account (24) and $\tau_1(q) = \phi_2(q,0)$, $\tau_2(q) = \phi_2(0,q)$ from (29) we will find $\tau_j(t)$ (j=1,2).

Hence, after having found $\tau_1(x)$ and $\nu_1(x)$, $\tau_2(y)$ and $\nu_2(y)$, the solution of the **Problem A** can be restored in the domain of Ω_0 as the solution of the **Problem N** (20), and in domains Δ_j (j=1,2) as the solution of the Cauchy problem. The solution of the **Problem A** in domains of D_j(j=1,2) can be restored as a solution of the Gaursat problem with conditions (6) and $u(x,y)\big|_{B_jE_j} = h_j(t)$, where $h_j(t)$ (j=1,2) are traces of solution of the Cauchy problems in domains Δ_j (j=1,2), on the line $x+y=q$, and reciprocally in domains Δ_j^* (j=1,2) as the solution of the Cauchy-Gaursat problem with conditions $u_y(x,0) = \nu_1^*(x)$; $u_x(0,y) = \nu_2^*(y)$, $-1 < x,y < -q$ and $u(x,y)\Big|_{B_j^*E_j^*} = h_j^*(t)$, where $h_j^*(t)$ (j=1,2) are traces of solution of the Gaursat problems in domains D_j (j=1,2). And finally the solution of the **Problem A** can be restored in the domain of Ω_0^* as the solution of the problem N, similarly as (20). The **Theorem** is proved.

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